PLANES OF PAIRS

The Tapestry of Patterns of Factors of Outputs of $x^2 + x + 41$

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PLANES OF PAIRS

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Introduction

Paul's Assignment

It was early 1987. A programmer at our office whom I was helping strengthen his Assembler language skills asked me to assign him a program to test what he had learned. I thought about it. I wanted to give Paul something very simple in concept yet sufficiently challenging technically. Then it came to me. A program that would meet both criteria and that would also achieve practical results would be one that generates triangular number arrays as detailed in my unpublished paper "Lines of Primes!".

Triangular Arrays

These triangular arrays are akin to the arrangement of numbered bowling pins or pocket pool balls in possible initial setups for those sports, viz.:

```
         1                                     1
      2     3                               2     3
   4     5     6          and            4     5     6         respectively.
7     8     9    10                   7     8     9    10
                                   11    12    13    14    15
```

Figure 1. Triangular arrays of bowling pins and pocket pool balls.

Originally just done this way, only extending for row upon row, the arrays evolved to being "left-justified" as in:

```
1
2  3
4  5  6
7  8  9  10
   etc.
```

Figure 2. Left-justified triangular array of whole numbers.

Before briefly describing variants of strictly triangular arrays that were also discussed in the original paper, an explanation of just what these arrays are used for is in order.

Prime Numbers

The triangular arrays and their variants are very useful for depicting patterns of prime numbers. As a reminder, primes are whole numbers not evenly divisible by any whole numbers other than themselves and 1. For example, while 23 is prime, since only 1 and 23 divide cleanly into it, 6 is not prime, being divisible by 2 and 3 besides by 1 and 6. The first several primes are: 2, 3, 5, 7, 11, 13, 17, 19, 23, and so on.

Positions in an array that correspond to primes are marked, while positions of non-primes are left blank. This makes the primes stand out visually.

Wilson's Theorem

In general, the prime numbers are not patterned. Though simple to understand, they defy attempts to derive a formula that will generate them. There is a formula, Wilson's Theorem, that will tell you if a number is prime, but it quickly becomes unwieldy for a person to use. Wilson's Theorem, that $(x - 1)! + 1$ is evenly divisible by x if and only if x is prime, involves the explosively large numbers called factorials.

Factorials

x factorial, or x!, is the product of all whole numbers from 1 to x. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. 10! is a 7-digit number. 20! is a 19-digit number. 50! is a 65-digit number. Any but the most trivial factorials are numbers unmanageably large, even for computers.

Patterns of Primes

There are many properties of the sequence of prime numbers that have been studied, yet patterns of primes have remained elusive. In my earlier work with triangular arrays I found quite a few patterns, as documented in "Lines of Primes!". By varying the counting scheme from one where the array has a single number ("1") in the first row and exactly one more number per row in each succeeding row, various patterns of primes appear. For example, by having two numbers in the top row and having each row contain two more numbers than the row preceding it, different patterns emerge than those in the original array.

The "lines" of primes found so far with the pictorial method are transitory, going only so far before being broken. Indeed, even with these inroads into finding pattern to them, the primes retain their mystique and aura of baffling randomness, given the simplicity of their concept. Yet, study of the coexistence of pattern and randomness in the primes may yield insight into natural processes and contribute to the discussion of a topic of increasing current interest, "the onset of chaos."

The Onset of Chaos

The new area of mathematics called "chaos theory" is concerned with describing natural phenomena exhibiting chaotic behavior, which until now has defied precise description. The approach is to look for transitions from order into chaos, such as in smooth flows becoming turbulent.

Patterns with Line Continuation of $x^2 + x + 41$

Returning to our events of 1987, while Paul was generating triangular arrays and tables of primes, it dawned on me that the work that I had done earlier on "patterns with line continuation" might lead to an ideal illustration of the onset of chaos.

Briefly, when many of the initially unbroken sequences of primes in the triangular arrays are finally broken, they do not immediately degenerate into complete chaos. For a short transition period there is pattern still. The following example will help to explain this. For the formula $x^2 + x + 41$, forty consecutive x's (zero through 39) yield a prime number as output. Then the streak is broken. But it is broken in the following way. The next two x's yield non-primes. The next two x's yield primes. Then comes a non-prime, 4 primes, a non-prime, 6 primes, a non-prime, 8 primes, a non-prime, and finally 10 primes. The following table summarizes this with more immediacy:

x's input to $x^2 + x + 41$	Consecutive Primes	Consecutive Non-primes
0 - 39	40	
40 - 41		2
42 - 43	2	
44		1
45 - 48	4	
49		1
50 - 55	6	
56		1
57 - 64	8	
65		1
66 - 75	10	
76 - ∞	apparent random distribution	

Table 1. Transition of $x^2 + x + 41$ from order to chaos.

Given a plot of primes on a triangular array with a top row of two numbers and succeeding rows with two more numbers than their predecessor rows, the $x^2 + x + 41$ outputs form a vertical column. With such a visual aid, it is easy to see the initial unbroken streak of 40 straight prime outputs going down, the stretch with the 2-4-6-8-10 pattern below that, and the random distribution of primes and non-primes from there on down. Such a transition from perfect order, through partial order, into complete chaos could well provide an ideal "laboratory" for studying the onset of chaos. I therefore asked Paul to prepare a listing of this variant of the triangular array but extended to x = 100,000.

Array Printout for x up to 100,000

Given this data to work with, I could easily look for any otherwise hidden patterns in the outputs of $x^2 + x + 41$. After eagerly scanning the listing to find nothing remarkable in the strings of primes revealed, I decided to shift my orientation to look at the outputs of $x^2 + x + 41$ that are *not* prime.

Non-prime Outputs

These non-primes are by definition "composite," being "composed of" or divisible by whole numbers besides themselves and 1. Might there be anything interesting about the divisors, or "factors," of the non-prime outputs? Another approach to predicting the prime outputs of $x^2 + x + 41$ might be predicting the non-prime outputs and thereby, indirectly, the primes.

Order in Chaos' Own Home Domain?

Although exploration of the territory well beyond the order-to-chaos transition zone discussed above might not immediately shed any new light on the onset of chaos, if it turned out that the home territory of chaos, the vast majority of the domain of outputs, were actually amenable to orderly analysis, then this might lead to insight into other application areas where chaos has *appeared* unbreachable. Even if no such orderliness were discovered within the chaos, it might still be rewarding to look there for whatever might be found.

As it turns out, the non-prime outputs of $x^2 + x + 41$ all fall into an exquisitely patterned tapestry when they are each decomposed into two factors.

Factors of Composite $x^2 + x + 41$ Outputs

The First Composites' Factors

The first composite $x^2 + x + 41$ for an x above zero is for x = 40. Calling $x^2 + x + 41$ the function of x, or f(x), f(40) = 1681. Besides being 1 • 1681, 1681 = 41 • 41. The next composite f(x) is f(41). f(41) = 1763 = 41 • 43. Next follow f(44) = 2021 = 43 • 47, f(49) = 2491 = 47 • 53, and so on as listed below in Table 2:

x	f(x)	Factors of f(x)
40	1681	41 • 41
41	1763	41 • 43
44	2021	43 • 47
49	2491	47 • 53
56	3233	53 • 61
65	4331	61 • 71
76	5893	71 • 83

Table 2. Factors of the first several composite $x^2 + x + 41$'s.

The numbers appearing here as factors are not just any numbers. They are the outputs of $x^2 + x + 41$ for the first several x's. f(0) = 41, f(1) = 43, and so on as in Table 3 below:

x	f(x)
0	41
1	43
2	47
3	53
4	61
5	71
6	83

Table 3. Outputs of $x^2 + x + 41$ for the first several x's.

$f(x) \cdot f(x+1)$ Composites

The patterning of factors seen in Table 2 suggests that in general $f(x) \cdot f(x + 1)$ yields a number that is also an output of $x^2 + x + 41$. This pattern does indeed continue indefinitely. One might question the very first factor pair, $41 \cdot 41$, as belonging in the pattern. This pair can be interpreted as $f(-1) \cdot f(0)$, since $f(-1)$ is also 41:

$$(-1)^2 + (-1) + 41 = 1 + (-1) + 41 = 41.$$

Thus the pattern holds for the first seven composite outputs, covering the zone of transition to chaos, x's from 40 through 76. Can we prove that $f(x) \cdot f(x + 1)$ always yields $f(y)$ where y is whole? Yes. By combining the information in Tables 2 and 3, we can derive the following:

y	f(y)	factors of f(y)	f(x)	f(x+1)	x	x+1	$(x+1)^2$	$40 + (x+1)^2$
40	1681	41•41	41	41	-1	0	0	40
41	1763	41•43	41	43	0	1	1	41
44	2021	43•47	43	47	1	2	4	44
49	2491	47•53	47	53	2	3	9	49
56	3233	53•61	53	61	3	4	16	56
65	4331	61•71	61	71	4	5	25	65
76	5893	71•83	71	83	5	6	36	76

Table 4. $f(x) \cdot f(x + 1)$ yields $f(y)$ for $x^2 + x + 41$.

Note that $y = 40 + (x + 1)^2$. Therefore:

$y = 40 + (x^2 + 2x + 1) = x^2 + 2x + 41$. So,

$f(y) = (x^2 + 2x + 41)^2 + (x^2 + 2x + 41) + 41.$

We need to show that this expression is equivalent to

$$f(x) \cdot f(x + 1) = (x^2 + x + 41) \cdot [(x + 1)^2 + (x + 1) + 41].$$

$$f(y) = (x^4 + 4x^3 + 86x^2 + 164x + 1681) + (x^2 + 2x + 41) + 41$$

$$= x^4 + 4x^3 + 87x^2 + 166x + 1763.$$

$$f(x) \cdot f(x+1) = (x^2 + x + 41) \cdot [(x^2 + 2x + 1) + (x + 1) + 41]$$
$$= (x^2 + x + 41) \cdot (x^2 + 3x + 43)$$
$$= x^4 + 4x^3 + 87x^2 + 166x + 1763.$$

Thus, $f(x) \cdot f(x + 1)$ always does yield some $f(y)$.

Immediate Exceptions

One might be tempted to conclude that all composite outputs factor in this way. Unfortunately, matters are not that simple. The very next composite output is $f(81) = 6683 = 41 \cdot 163$. In fact, before we hit the next output of type $f(x) \cdot f(x + 1)$, where the factors are $83 \cdot 97 = 8051 = f(89)$, we find $f(82) = 6847 = 41 \cdot 167$, $f(84) = 7181 = 43 \cdot 167$, and $f(87) = 7697 = 43 \cdot 179$.

Systematic List of Factor Pairs

Systematically listing the factor pairs for all of the composite outputs for $x = 40$ up to $x = 190$, we can see the beginnings of a fascinating tapestry of interwoven sequences of factor pair patterns:

x	f(x)	Factors
40	1681	41 • 41
41	1763	41 • 43
44	2021	43 • 47
49	2491	47 • 53
56	3233	53 • 61
65	4331	61 • 71
76	5893	71 • 83
81	6683	41 • 163
82	6847	41 • 167
84	7181	43 • 167
87	7697	43 • 179
89	8051	83 • 97
91	8413	47 • 179
96	9353	47 • 199
102	10547	53 • 199
104	10961	97 • 113
109	12031	53 • 227
117	13847	61 • 227
121	14803	113 • 131
122	15047	41 • 367
123	15293	41 • 373
126	16043	61 • 263
127	16297	43 • 379
130	17071	43 • 397
136	18673	71 • 263
138	19223	47 • 409
140	19781	131 • 151
143	20633	47 • 439
147	21797	71 • 307
155	24221	53 • 457
159	25481	83 • 307
161	26123	151 • 173
162	26447	53 • 499
163	26773	41 • 653
164	27101	41 • 661
170	29111	43 • 677
172	29797	83 • 359
173	30143	43 • 701
178	31903	61 • 523
184	34081	73 • 197
185	34451	47 • 733
186	34823	97 • 359
187	35197	61 • 577
190	36331	47 • 773

Table 5. Interwoven sequences of factor pair patterns.

Tapestry Mapping

In order to get a handle on the scheme of the composite outputs of $x^2 + x + 41$, if such a scheme exists, the tapestry illustrated in Table 5 must be understood. The next logical step in unraveling this increasingly intricate sequence of threads is to break the single column of factor pairs into individual columns for each unique subsequence within it. For example, the subsequence starting with 41 • 41 would appear in one column, while the subsequence starting with 41 • 163 would appear in a second column to the right of the first column. The results of this approach when applied to the entries in Table 5, along with some additional data from beyond x = 190, appear in Table 6:

x	f(x)	Seq.1	Seq.2	Seq.3	Seq.4	Seq.5	Seq.6	Seq.7
40	1681	41• 41						
41	1763	41• 43						
44	2021	43• 47						
49	2491	47• 53						
56	3233	53• 61						
65	4331	61• 71						
76	5893	71• 83						
81	6683		41•163					
82	6847		41•167					
84	7181		43•167					
87	7697		43•179					
89	8051	83• 97						
91	8413		47•179					
96	9353		47•199					
102	10547		53•199					
104	10961	97•113						
109	12031		53•227					
117	13847		61•227					
121	14803	113•131						
122	15047			41•367				
123	15293			41•373				
126	16043		61•263					
127	16297			43•379				
130	17071			43•397				
136	18673		71•263					
138	19223			47•409				
140	19781	131•151						
143	20633			47•439				
147	21797		71•307					
155	24221			53•457				
159	25481		83•307					
161	26123	151•173						
162	26447			53•499				
163	26773				41•653			
164	27101				41•661			
170	29111				43•677			
172	29797		83•359					
173	30143				43•701			
178	31903			61•523				
184	34801	173•197						
185	34451				47•733			
186	34823		97•359					
187	35197			61•577				
190	36331				47•773			
201	40643		97•419					

204	41861					41•1021		
205	42271					41•1031		
207	43097			71•607				
208	43513				53•821			
209	43931	197•223						
213	45623					43•1061		
215	46481				53•877			
216	46913					43•1091		
217	47347		113•419					
218	47783			71•673				
232	54097					47•1151		
234	55031		113•487					
236	55973	223•251						
237	56447					47•1201		
239	57401				61•941			
242	58847			83•709				
244	59821							163•367
245	60311						41•1471	
246	60803						41•1483	
248	61793				61•1013			
249	62291							167•373
251	63293							167•379
252	63797		131•487					
255	65321			83•787				
256	65833						43•1531	
259	67381						43•1567	
261	68423					53•1291		
265	70531	251•281						
266	71063							179•397
268	72133					53•1361		
270	73211							179•409
271	73753		131•563					
278	77603				71•1093			
279	78161						47•1663	
283	80413			97•829				
284	80981						47•1723	

Table 6. Unique factor pair sequences split into separate columns.

Pattern Across Subsequences

If one looks at the x's up through around 240, all of the composite outputs appear to fall in one or another of the subsequences beginning with 41 • "something" and 41 • "something a little higher." And, these "somethings" themselves all fall into a nice pattern:

Subsequence or Series	First Factor Pair in Series	Second Factor	In Terms Of 41
1	41 • 41	41	1 • 41 − 0
2	41 • 163	163	4 • 41 − 1
3	41 • 367	367	9 • 41 − 2
4	41 • 653	653	16 • 41 − 3
5	41 • 1021	1021	25 • 41 − 4

Table 7. Pattern of series-starting factor pairs' second factor.

So, these "series starters" are, for series x, equal to 41 • $[x^2 \cdot 41 - (x - 1)] = 41 \cdot (41x^2 - x + 1)$. The second factor pair in each of these series consists of 41 and $[x^2 \cdot 41 - (x - 1)] + 2x = 41x^2 + x + 1$. For example, the second factor pair in the series begun by the factor pairs in Table 7 are:

Subsequence or Series	Second Factor Pair in Series	Second Factor	Second Factor From First Factor Pair	Difference
1	41 • 43	43	41	2
2	41 • 167	167	163	4
3	41 • 373	373	367	6
4	41 • 661	661	653	8
5	41 • 1031	1031	1021	10

Table 8. Relationship between first and second factor pair.

Exceptions to 41-type Sequences

Again, "unfortunately," this pattern of series alone cannot account for all of the composite outputs. The first indication that this pattern is not all that there is occurs at f(x) for x = 244. This number, 59821, is 163 • 367. None of the 41-type subsequences contain this factor pair. Several more "anomalous" factor pairs quickly turn up as well, nearby. f(249) = 62291 = 167 • 373, f(251) = 63292 = 167 • 379, and so on.

163-type Sequences?

It seems that a new type of series, with the first factor of its first factor pair equal to 163 instead of 41, begins with f(244). I naturally sought other 163-based sequences. Since 163 was familiar from Table 7, I tried 163 • 653, 163 • 1021, and so on, to see if their products were outputs of $x^2 + x + 41$. They are not. Yet I felt sure that there should be a whole family of series based on 163 just as there is one based on 41.

Other-type Sequences?

For that matter, now that the precedent was set where 41 did not start all series, I also looked at 367 • 653, 653 • 1021, and so on, as series starters in analogy with 163 • 367, given Table 7. But what other 163-based, 367-based, and so on, series were there, if any, as it seemed there must be?

Printout for x up to A Million

At this point I asked Paul to generate the array for numbers up to a million. I needed more data.

Anomalies

When the array was produced, I made a systematic chart of all composite outputs for f(x) up to a million. In the chart, every 41•n -based composite was marked as such, and every 163 • 367 -based composite, 367 • 653 -based composite, and so on, was also marked. Any composite not accounted for by any of these series must therefore belong to some as yet unknown set. The "anomalies" that turned up, providing me with priceless data to work with, are as follows:

x	Factors of f(x)	x	Factors of f(x)	x	Factors of f(x)
407	163 • 1019	630	379 • 1049	835	499 • 1399
416	167 • 1039	652	263 • 1619	856	367 • 1999
418	167 • 1049	662	263 • 1669	867	523 • 1439
445	179 • 1109	663	397 • 1109	869	373 • 2027
449	179 • 1129	679	409 • 1129	885	379 • 2069
494	199 • 1229	693	199 • 2417	892	199 • 4003
500	199 • 1259	699	199 • 2459	896	163 • 4931
563	227 • 1399	733	163 • 3301	898	199 • 4057
570	163 • 1997	734	439 • 1229	915	263 • 3187
571	227 • 1439	752	167 • 3391	924	397 • 2153
583	167 • 2039	758	457 • 1259	925	263 • 3257
585	167 • 2053	761	307 • 1889	956	409 • 2237
611	367 • 1019	773	307 • 1949	966	577 • 1619
622	373 • 1039	790	227 • 2753	978	367 • 2609
624	179 • 2179	803	179 • 3607	986	179 • 5437
628	179 • 2207	807	179 • 3643	995	373 • 2657

Table 9. "Anomalies" -- factor pairs not fitting known series.

Teasing out the Threads

Again, it is productive to separate this mixed list into subsequences, each with its own column. A condensed form of that is given in Table 10 as follows:

x	Factors of f(x)	x	Factors of f(x)	x	Factors of f(x)
407	163 • 1019	570	163 • 1997	733	163 • 3301
416	167 • 1039	583	167 • 2039	(750	167 • 3373)
418	167 • 1049	585	167 • 2053	752	167 • 3391
445	179 • 1109	624	179 • 2179	803	179 • 3607
449	179 • 1129	628	179 • 2207	807	179 • 3643
494	199 • 1229	693	199 • 2417	892	199 • 4003
500	199 • 1259	699	199 • 2459	898	199 • 4057
563	227 • 1399	790	227 • 2753		
571	227 • 1439	(798	227 • 2809)		
652	263 • 1619	915	263 • 3187		
662	263 • 1669	925	263 • 3257	896	163 • 4931
761	307 • 1889			(917	167 • 5041)
773	307 • 1949			(919	167 • 5063)
(890	359 • 2209)			982	179 • 5393
(904	359 • 2279)			986	179 • 5437
611	367 • 1019	856	367 • 1999	978	367 • 2609
622	373 • 1039	869	373 • 2027	995	373 • 2657
630	379 • 1049	885	379 • 2069		
663	397 • 1109	924	397 • 2153		
679	409 • 1129	956	409 • 2237		
734	439 • 1229				
758	457 • 1259				
835	499 • 1399				
867	523 • 1439				
966	577 • 1619				

Table 10. The anomalies organized.

Note: The list entries in parentheses were not "anomalies," their product being producible by an already known series such as the series beginning with 41 • 163.

Families of Series

The family of 41•n -based series was now complemented by families of series based on 163•n and 367•n as follows:

41 • 41	163 • 367	367 • 653
41 • 163	163 • 1019	367 • 1019
41 • 367	163 • 1997	367 • 1999
41 • 653	163 • 3301	367 • 2609
41 • 1021		

Table 11. Three families of series-starting factor pairs.

A Brief Look Ahead

We are now about to embark on a journey into the details of these known families of series of factor pairs. Along the way we will derive a similar family of series for 653. In the process we will develop and use several parameters for analyzing a family. This analysis will reshape the 41-family. This will in turn reshape the other families.

By the time that we emerge again, we will have a powerful arsenal of analytical tools and a deeper understanding of these families. We will then be ready to return to mapping out the families. We will see how the families are themselves grouped into a superfamily, or plane.

That is enough looking ahead for now, but I will nevertheless mention that our travels will eventually take us quite far beyond even the level of the plane of families.

First Family Portraits

Summary Charts

A summary chart was prepared for each of the three known families of series. Each chart contains several aspects. Please refer to Chart 1 in Appendix A during the following discussion.

The charts are arranged into columns by series, and into rows by aspect. Above each column is a series label, for example ${}^{1}{}_{3}S$ for the series starting with factor pair 41 • 367.

Series Designation

In general, the notation ${}^{v}{}_{w}S_{x}$ refers to the x^{th} factor pair in series w within family v. So, ${}^{1}{}_{3}S_{6}$ is the sixth factor pair in the third series in the first family. The first family is the 41-family; the third series within the 41-family is the one beginning with 41•367; the sixth factor pair in that series is 47•439.

Chart Aspects

Moving down the left-hand side of a chart, the aspect labels are respectively:

x_1
operands
x spacing
op2 spacing
$op2_1:41$
$x_1:41$
op1 spacing

Each of these will be explained in turn.

x_1

The two numbers in a factor pair produce another number when multiplied together. Our factor pairs produce products equal to outputs of $x^2 + x + 41$. For example, 41•163 = 6683. $6683 = x^2 + x + 41$ for x = 81. 81 is thus the input x into $x^2 + x + 41$ that produces 6683, which has the factor pair 41•163. In our summary charts, "x_1" refers to the x input for the first factor pair in a factor pair series. Thus, in this example x_1 would be 81. x_2 would refer to the second series-member's input value -- in the case of the series started by 41•163 the second factor pair is 41•167, which produces 6847. The input x that produces 6847 is 82 (82^2 + 82 + 41 = 6847), so x_2 for this series is 82. x_3 would be 84 in this case, since f(84) = 7181 = 43•167, the third factor pair in this factor pair series. And so on.

Operands

The term "operands" here is used synonymously with "factor pair," in this case the factor pair starting a series. In general, "op1" is the left-hand factor in a factor pair, while "op2" is the right-hand factor. For series ${}^{1}{}_{4}S$, the starting pair is 41•653, so the operands are 41 and 653.

x Spacing

This is the spacing between x's in the series. Table 12 below shows the first eight entries in the series begun by 41•367:

x	f(x)	operands
122	15047	41 • 367
123	15293	41 • 373
127	16297	43 • 379
130	17071	43 • 397
138	19223	47 • 409
143	20633	47 • 439
155	24221	53 • 457
162	26447	53 • 499

Table 12. The series begun by 41 • 367.

Odd and Even Spacers

It turns out that the spacing between consecutive x's follows a simple pattern:

x	Distance Between x's			
122				
	1	=	1•1	
123				
	4	=		2•2
127				
	3	=	1•3	
130				
	8	=		2•4
138				
	5	=	1•5	
143				
	12	=		2•6
155				
	7	=	1•7	
162				

Table 13. The pattern of spacing between consecutive x's.

This pattern for series spacing, of one coefficient (multiplier) applying to all odd numbers and another coefficient applying to all even numbers, comes up everywhere in this work. We can abbreviate the spacing in such a well-behaved series by simply noting the odd numbers' coefficient and the even numbers' coefficient. For example, for ${}^{1}{}_{3}S$ the x spacing is indicated as:

1•1,3,5...	or just	1•
2•2,4,6...		2•

A, B, C, and D

We will introduce four items, named A_n, B_n, C_n, and D_n, which help with calculating values in these series. Some series are very simple, having the form

x	Distance Between x's	
x_1		
	$\#_1$	
x_2		
		$\#_2$
x_3		
	$\#_1$	
x_4		
		$\#_2$
x_5		
	$\#_1$	
x_6		
etc.		

Table 14. The pattern of distance between x's.

so that x_6, for example, is equal to $x_1 + \#_1 + \#_2 + \#_1 + \#_2 + \#_1 = x_1 + 3 \cdot \#_1 + 2 \cdot \#_2$

Floor and Ceiling

The way to summarize this pattern is to find how many $\#_1$'s and $\#_2$'s to add to x_1 for any given n, to get x_n.

Looking at the following table it can be seen that the number of $\#_1$'s to add for n is n÷2 rounded down. The number of $\#_2$'s to add for n is n÷2 rounded up, minus one. A number x rounded down, called the "floor" of x, will be expressed as $\lfloor x \rfloor$. Similarly, a number x rounded up, called the "ceiling" of x, will be represented here as $\lceil x \rceil$.

n	x_n	Number of $\#_1$'s	Number of $\#_2$'s
1	x_1	0	0
2	$x_1 + \#_1$	1	0
3	$x_1 + \#_1 + \#_2$	1	1
4	$x_1 + \#_1 + \#_2 + \#_1$	2	1
5	$x_1 + \#_1 + \#_2 + \#_1 + \#_2$	2	2
6	$x_1 + \#_1 + \#_2 + \#_1 + \#_2 + \#_1$	3	2

Table 15. How many times to include odd and even spacing for x.

We can thus summarize as follows:

$$x_n = x_1 + (\lfloor n \div 2 \rfloor \cdot \#_1) + [(\lceil n \div 2 \rceil - 1) \cdot \#_2]$$

Example for Simple First-power Spacing

To illustrate that this really works, let us use an example. We will first build a series of this type, then demonstrate that the formula produces the equivalent result.

Let us take $x_1 = 100$, $\#_1 = 5$, and $\#_2 = 10$. This means that the first number in the series is 100, the second is 105, the third is 115, the fourth is 120, and so on as follows:

n	x_n	$\#_1$	$\#_2$
1	100		
		5	
2	105		
			10
3	115		
		5	
4	120		
			10
5	130		
		5	
6	135		

Table 16. An example showing odd and even spacing.

Now, let us see how the formula gives the same answers. "Sum" here indicates the sum of three things -- x_1, $\lfloor n \div 2 \rfloor \cdot \#_1$, and $(\lceil n \div 2 \rceil - 1) \cdot \#_2$ -- in other words 100 plus the two columns preceding "sum."

n	x_n	$n \div 2$	$\lfloor n \div 2 \rfloor$	$\lceil n \div 2 \rceil$	$\lceil n \div 2 \rceil - 1$	$\#_1$	$\#_2$	$\lfloor n \div 2 \rfloor \cdot \#_1$	$(\lceil n \div 2 \rceil - 1) \cdot \#_2$	sum
1	100	.5	0	1	0	5	10	0	0	100
2	105	1.0	1	1	0	5	10	5	0	105
3	115	1.5	1	2	1	5	10	5	10	115
4	120	2.0	2	2	1	5	10	10	10	120
5	130	2.5	2	3	2	5	10	10	20	130
6	135	3.0	3	3	2	5	10	15	20	135

Table 17. Verifying that the formula using odd and even spacing is correct.

A_n and B_n

For ease of notation, we can call $\lfloor n \div 2 \rfloor$ "A_n" and $\lceil n \div 2 \rceil -1$ "B_n." Thus:

$$x_n = x_1 + \lfloor n \div 2 \rfloor \cdot \#_1 + (\lceil n \div 2 \rceil -1) \cdot \#_2 \text{ becomes}$$

$$x_n = x_1 + A_n \cdot \#_1 + B_n \cdot \#_2.$$

Second-power Spacing

In the slightly more complicated series such as ${}^1{}_3S$ that have the form illustrated in Table 13, the formula for x_n is:

$$x_n = x_1 + \lfloor n \div 2 \rfloor^2 \cdot \#_1 + \lceil n \div 2 \rceil (\lceil n \div 2 \rceil -1) \cdot \#_2$$

This is equivalent to:

$$x_n = x_1 + A_n^2 \cdot \#_1 + (B_n+1)(B_n) \cdot \#_2$$

C_n and D_n

Again for convenience, we will use "C_n" for A_n^2, and "D_n" for $(B_n+1)(B_n)$. So

$$x_n = x_1 + C_n \cdot \#_1 + D_n \cdot \#_2.$$

Generating x's By Formula

Now we can look at the x's for a factor pair series such as ${}^1{}_3S$ and see how we can generate any given member by using the formula:

n	x_n	x_1	$A_n = \lfloor n \div 2 \rfloor$	$B_n = \lceil n \div 2 \rceil -1$	$C_n = A_n^2$	$D_n = (B_n+1)(B_n)$	$\#_1$	$\#_2$	$C_n \cdot \#_1$	$D_n \cdot \#_2$	sum
1	122	122	0	0	0	0	1	2	0	0	122
2	123	122	1	0	1	0	1	2	1	0	123
3	127	122	1	1	1	2	1	2	1	4	127
4	130	122	2	1	4	2	1	2	4	4	130
5	138	122	2	2	4	6	1	2	4	12	138
6	143	122	3	2	9	6	1	2	9	12	143
7	155	122	3	3	9	12	1	2	9	24	155
8	162	122	4	3	16	12	1	2	16	24	162

Table 18. Generating ${}^1{}_3S$ x's using the second power x-spacing formula.

Given the formula we could now directly find any member in the series of x's.

op2 Spacing

Getting back to Chart 1 in Appendix A, "op2 spacing" is the spacing between consecutive operand2's, or right-hand factors in the factor pairs in a factor pair series. op2 spacings are also of the form $x_1 + C_n \cdot \#_1 + D_n \cdot \#_2$, yielding $op2_n$ as follows:

$op2_n = op2_1 + C_n \cdot \#_1 + D_n \cdot \#_2$.

Similarly, all that we need to denote op2 spacing is to list $\#_1$ and $\#_2$: $\#_1\cdot$, $\#_2\cdot$.

$op2_1$:41

"$op2_1$:41" is an operand2 in the first factor pair in a series, in terms of how many times 41 it is, and if not exactly a multiple of 41, how far from such a multiple it is. For example, for Chart 1 the $op2_1$'s are 41, 163, 367, 653, 1021, and 1471. These are, respectively, 1•41-0, 4•41-1, 9•41-2, 16•41-3, 25•41-4, and 36•41-5.

x_1:41

"x_1:41" concerns x_1 in terms of 41 in the same manner as for $op2_1$:41.

op1 Spacing

"op1 spacing" is the spacing between successive op1's in a factor pair series. These too are of the same form as for x spacing and are denoted the same way: $\#_1\cdot$, $\#_2\cdot$. There is only one op1 spacing entry for a whole chart, however, since the same op1 sequence applies to all columns in a chart.

Patterns Across Series

Looking at Chart 1, note the patterns in each row. For the 41-family, in the x spacing rows, the odd coefficients are all 1, while the even coefficients increase by 1 with each series. Also, the sum of each such coefficient-pair forms a pattern -- in this case the numbers again increase by 1 per series.

For 41-series n, the op2 spacing coefficients are 2n and n^2-2n for the "odd" vs. the "even" coefficients, respectively. The sum of the odd and the even coefficients for 41-series n is n^2.

$op2_1$:41 for each 41-series is n^2•41-(n-1) and x_1:41 is n•41-1.

Finally, op1 spacing is 0•, 1• with sum = 1, for all of the 41-series.

The New ^1_1S

Of particular note are the x spacing, op2 spacing, and op1 spacing values for series ^1_1S, the 41-series starting with the factor pair 41 • 41. Why? Because these are not the values that would be applicable to the series as it appears in Table 6. The values that we do show in Chart 1 were derived by extrapolation from those of the other 41-series.

If Table 6 were used to derive the ^1_1S values for x spacing, op2 spacing, and op1 spacing, then those values would be:

x	^1_1S as in Table 6	x spacing	op2 spacing	op1 spacing
40	41 • 41			
		1	2	0
41	41 • 43			
		3	4	2
44	43 • 47			
		5	6	4
49	47 • 53			
		7	8	6
56	53 • 61			
		9	10	8
65	61 • 71			
		11	12	10
76	71 • 83			

Table 19. ^1_1S spacings if Table 6 were used to generate them.

These spacings in terms of odd and even coefficients as we have been using them are:

x spacing		op2 spacing		op1 spacing	
1÷1 • 1		2 • 1		0÷1 • 1	
	3÷2 • 2		2 • 2		2÷2 • 2
5÷3 • 3		2 • 3		4÷3 • 3	
	7÷4 • 4		2 • 4		6÷4 • 4
9÷5 • 5		2 • 5		8÷5 • 5	
	11÷6 • 6		2 • 6		10÷6 • 6

Table 20. Odd and even coefficients for Table 19 spacings.

As can be seen, the only one of these three coefficient pairs with any chance of conforming to the format that we have seen everywhere else, of whole number and constant coefficients for both odd and even spacings, is the 2•, 2• for op2 spacing.

op2 Spacings in Family 1

Putting aside for the moment that the x spacing and op1 spacing are not even of the desired format, how does the op2 spacing coefficient pair fit in with those of the other 41-series?

The odd coefficients of op2 spacing for the 41-series above ^1_1S are: 4, 6, 8, 10, and 12. So, if ^1_1S's odd coefficient is 2, then that fits the pattern. But, an even coefficient of 2 does not fit with the succeeding values of 0, 3, 8, 15, and 24. What *does* fit is -1.

Extrapolated ^1_1S Spacings

So, by extrapolation, then, let us say that op2 spacing for ^1_1S is 2•, -1•. What about the x spacing and op1 spacing? The values for x spacing in the rest of the 41-series dictate that ^1_1S's x spacing should be 1•, 0• Similarly, the op1 spacing should be 0•, 1•. But, given these coefficient pairs for op2 spacing, x spacing, and op1 spacing, what happens to ^1_1S? What is its new list of factor pairs, governed by these new spacings? Do these new factor pairs still somehow make sense?

The Adjusted ^1_1S Sequence

Still starting with 41•41 as the first factor pair in the series, the "new" version of ^1_1S looks like this:

x	$^{1}_{1}S$	x spacing	op2 spacing	op1 spacing
40	41•41			
		1•1	2•1	0•1
41	41•43			
		0•2	−1•2	1•2
41	43•41			
		1•3	2•3	0•3
44	43•47			
		0•4	−1•4	1•4
44	47•43			
		1•5	2•5	0•5
49	47•53			
		0•6	−1•6	1•6
49	53•47			

Table 21. The adjusted spacings for $^{1}_{1}S$.

The x's are the familiar ones, but occurring twice (except for x = 40). Similarly, the factor pairs are the familiar ones, occurring twice, once "forward" and once "reversed." Nothing correct is missing, nor is anything spurious present. Why doesn't x = 40 appear twice? Actually, it does.

How x = 40 Does Appear Twice

Extending the x spacing, op2 spacing, and op1 spacing sequences back another step from x = 40 yields:

x	$^{1}_{1}S$	x spacing	op2 spacing	op1 spacing
40	41•41			
		0•0	−1•0	1•0
40	41•41			
		1•1	2•1	0•1
41	41•43			
		0•2	−1•2	1•2

Table 22. How x = 40 appears twice in $^{1}_{1}S$.

This makes the entry previous to x = 40 identical to it. Interestingly, to be consistent with the rest of the "new" series, the factor pair in our original x = 40 entry can be considered the "reverse" of the new x = 40's factor pair, even though both are 41•41.

Majority Rule

So, the "new" ${}^{1}{}_{1}S$, although "derived," is preferable to the "original" version because its internal attributes are consistent with those of all of the other 41-family series. Its values are still valid in the original context of a factor pair sequence for $x^2 + x + 41$ outputs, which is essential if the "new" series is to be anything more than a "laboratory" curiosity. The principle of what holds for the majority constraining what must hold for the exception is a powerful one for deciding which one of a choice of otherwise equally viable alternative interpretations should be selected. This principle is called into play often in this work.

${}^{2}{}_{0}S$

Let us now look at Charts 2 and 3 in the same way. Without listing here all of the row patterns, by looking at them in Chart 2 it becomes clear that there should be another column for "${}^{2}{}_{0}S$" on the left of ${}^{2}{}_{1}S$. The x spacing coefficients would lead, moving left from ${}^{2}{}_{1}S$, to 1•, 1•; the op2 spacing coefficients point to a pair with even-coefficient = 1 and coefficient-sum = 1^2, therefore a pair = 0•, 1•; the $op2_1$:41 values yield 1^2•41−0 = 41; the x_1:41 row extrapolates leftward to 2•41−1 = 81.

All of these extensions lead to ${}^{2}{}_{0}S$ being started by factor pair 163 • 41. This is indeed the same series as in 41-family series 2, ${}^{1}{}_{2}S$, with the operands reversed in each factor pair. In other words, instead of going 41•163, 41•167, 43•167, etc., here the series goes 163•41, 167•41, 167•43, etc. This explains why the op2 spacing here equals the op1 spacing there and vice versa.

${}^{3}{}_{0}S$

With Chart 3, a similar effect causes us to derive a column on the left edge, "${}^{3}{}_{0}S$," with x spacing = 1•, 2•; op2 spacing = 0•, 1•; $op2_1$:41 = 1^2•41−0 = 41; and x_1:41 = 3•41−1 = 122. This ${}^{3}{}_{0}S$ series starts with 367•41 and is the "reverse" of ${}^{1}{}_{3}S$, which starts with 41•367.

Spacings of Spacings

The patterns in the rows in Chart 3 are a little more complex than in Charts 1 and 2. For example, the sequence of x spacing coefficients is:

367-series	odd x spacing coefficient	even x spacing coefficient
1	5	1
2	7	5
3	11	4
4	13	8
5	17	7
6	19	11

Table 23. x spacing coefficients for the ^3S family.

Knowing the pattern of the spacings between *these* spacings allows us to extrapolate. Those spacings between spacings can be seen in the following:

367-series	odd x spacing coefficient	spacing	even x spacing coefficient	spacing
1	5		1	
		2		4
2	7		5	
		4		-1
3	11		4	
		2		4
4	13		8	
		4		-1
5	17		7	
		2		4
6	19		11	

Table 24. Pattern of ^3S family x spacings.

By extending these relationships to ^3_0S we can derive its attributes' values, for example for x spacing, as follows:

367-series	odd x spacing coefficient	spacing	even x spacing coefficient	spacing
0	1		2	
		4		-1
1	5		1	
		2		4
2	7		5	
		4		-1
3	11		4	

Table 25. Extending the ^3S x spacing pattern to series 0.

z

x_1:41 Coefficient vs. Sum of x Spacing Coefficients; op2_1:41 Coefficient vs. Sum of op2 Spacing Coefficients

Given the attributes of a series family as described so far in the charts, another aspect suggested itself. First, as listed in the following tables, note how for each chart the x_1•41 coefficient equals the sum of the x spacing coefficients for a column. Also note how not only are the op2_1:41 coefficient and op2 spacing coefficient sum for a column also equal, but that these values are in each case perfect squares as well:

			x spacing				op2 spacing	
Series	x_1:41	Coef.	Coef.'s	Sum	op2_1:41	Coef.	Coef.'s	Sum
${}^{1}_{0}S$	0•41-1	0	1, -1	0	0•41-(-1)	0	0, 0	0
${}^{1}_{1}S$	1•41-1	1	1, 0	1	1•41-0	1	2, -1	1
${}^{1}_{2}S$	2•41-1	2	1, 1	2	4•41-1	4	4, 0	4
${}^{1}_{3}S$	3•41-1	3	1, 2	3	9•41-2	9	6, 3	9
${}^{1}_{4}S$	4•41-1	4	1, 3	4	16•41-3	16	8, 8	16
${}^{1}_{5}S$	5•41-1	5	1, 4	5	25•41-4	25	10, 15	25
${}^{1}_{6}S$	6•41-1	6	1, 5	6	36•41-5	36	12, 24	36

Table 26. x_1:41, x spacing, op2_1:41, and op2 spacing coefficient patterns.

			x spacing				op2 spacing	
Series	x_1:41	Coef.	Coef.'s	Sum	op2_1:41	Coef.	Coef.'s	Sum
${}^{2}_{0}S$	2•41-1	2	1, 1	2	1•41- 0	1	0, 1	1
${}^{2}_{1}S$	6•41-2	6	5, 1	6	9•41- 2	9	6, 3	9
${}^{2}_{2}S$	10•41-3	10	9, 1	10	25•41- 6	25	20, 5	25
${}^{2}_{3}S$	14•41-4	14	13, 1	14	49•41-12	49	42, 7	49
${}^{2}_{4}S$	18•41-5	18	17, 1	18	81•41-20	81	72, 9	81
${}^{2}_{5}S$	22•41-6	22	21, 1	22	121•41-30	121	110, 11	121

Table 27. x_1:41, x spacing, op2_1:41, and op2 spacing coefficient patterns.

			x spacing				op2 spacing	
Series	x_1:41	Coef.	Coef.'s	Sum	$op2_1$:41	Coef.	Coef.'s	Sum
${}^3_0S$	3•41-1	3	1, 2	3	1•41- 0	1	0, 1	1
${}^3_1S$	6•41-2	6	5, 1	6	4•41- 1	4	4, 0	4
${}^3_2S$	12•41-3	12	7, 5	12	16•41- 3	16	8, 8	16
${}^3_3S$	15•41-4	15	11, 4	15	25•41- 6	25	20, 5	25
${}^3_4S$	21•41-5	21	13, 8	21	49•41-10	49	28, 21	49
${}^3_5S$	24•41-6	24	17, 7	24	64•41-15	64	48, 16	64
${}^3_6S$	30•41-7	30	19, 11	30	100•41-21	100	60, 40	100

Table 28. x_1:41, x spacing, $op2_1$:41, and op2 spacing coefficient patterns.

Matchings Related

Looking at just what these perfect squares are the squares of, it becomes clear that both of these match sequences for a chart, for example the 3, 6, 12, 15,... and 1, 4, 16, 25,... sequences of Table 28, are themselves intimately related. The numbers in the x_1:41 vs. x spacing match sequences are all divisible by a common factor. When the square root of the numbers in the $op2_1$:41 vs. op2 spacing match sequence has been taken, the resulting numbers are identical to those in the other match sequence after division by the common factor:

Series	x_1:41 Coef. or x Spacing Coef. Sum	÷1	$op2_1$:41 Coef. or op2 Spacing Coef. Sum	√
${}^1_0S$	0	0	0	0
${}^1_1S$	1	1	1	1
${}^1_2S$	2	2	4	2
${}^1_3S$	3	3	9	3
${}^1_4S$	4	4	16	4
${}^1_5S$	5	5	25	5
${}^1_6S$	6	6	36	6

Table 29. The relation between the two match sequences.

Series	x_1:41 Coef. or x Spacing Coef. Sum	÷2	$op2_1$:41 Coef. or op2 Spacing Coef. Sum	√
${}^{2}_{0}S$	2	1	1	1
${}^{2}_{1}S$	6	3	9	3
${}^{2}_{2}S$	10	5	25	5
${}^{2}_{3}S$	14	7	49	7
${}^{2}_{4}S$	18	9	81	9
${}^{2}_{5}S$	22	11	121	11

Table 30. The relation between the two match sequences.

Series	x_1:41 Coef. or x Spacing Coef. Sum	÷3	$op2_1$:41 Coef. or op2 Spacing Coef. Sum	√
${}^{3}_{0}S$	3	1	1	1
${}^{3}_{1}S$	6	2	4	2
${}^{3}_{2}S$	12	4	16	4
${}^{3}_{3}S$	15	5	25	5
${}^{3}_{4}S$	21	7	49	7
${}^{3}_{5}S$	24	8	64	8
${}^{3}_{6}S$	30	10	100	10

Table 31. The relation between the two match sequences.

The sequence that both match sequences share, for example 1, 2, 4, 5, 7, 8,... for the 367-family, is called "z." The number that x spacing coefficient sums for a family are divided by is simply the family number. This is called "v" as in ${}^{v}_{w}S$.

Is There a Family of Families?

Is There a 653-series Family?

A natural question, given the evidence for there being families of factor pair series for 41, 163, and 367, was "Are there an infinite number of such families?" If other such families could be found, then the likelihood of there being such a family of families would be strengthened.

The first logical step was to look for another family of series, one involving 653. The sequence of 41-series starting factor pairs goes 41•1, 41•41, 41•163, 41•367, 41•653, 41•1021, etc. Since 41, 163, and 367 all had families, 653 seemed likely to follow suit.

There were already a few components for a 653-family. 653•41 is simply a reversed version of the known series 41•653. 653•367 is the reversed version of 367•653. 653•1021 had been established earlier. What might other series be in a prospective 653-family?

From Array Construction to Pure Deduction

If it were simply a matter of generating enough further anomalies, then Paul's program could churn out more of the prime vs. composite visual points in $x^2 + x + 41$ and I could then laboriously determine which of the composites were generated by 41-, 163-, or 367-series, and which composites were not, and were therefore new anomalies. Some of those new anomalies might well be 653-series members. In general, any such anomalies found would be valuable data for determining any other series of factor pairs, possibly even new families of series.

The search for more data was not done in this way, however. I attempted to discover the new series and families through pure deduction rather than through data analysis. This was not because of the tedium of finding anomalies among familiar series' offspring, but simply because we could no longer run Paul's program due to our work climate limiting how much processing Paul was allowed. And, given success with a pure deduction approach, we would avoid the inevitable geometrically-growing tedium of hunting down anomalies in ever larger, more intricately populated samples of outputs of $x^2 + x + 41$.

Of course, the pure deduction approach does work with data somewhat, but often the data is the knowledge of whole series' attributes, rather than knowledge of individual factor pairs. Of course, also, the data-driven approach of generating "raw" data in the form of factor pairs, and then inducing (recognizing) patterns in the data, thus does work with deduction somewhat. There is an element of "generate and test" in the pure deduction approach as will be seen shortly. The difference between the two approaches is that in the pure deduction method both the data and the deduction (or induction) are at a higher level of abstraction. This higher level of operation can be vastly more efficient at getting one to general, powerful conclusions, besides saving much time-consuming raw data muck-wading.

The First Three 653-series

A first attempt to lay out how the 653-series might fit in a family is depicted as follows:

	${}^4_1S$	${}^4_2S$	${}^4_3S$	${}^4_4S$	${}^4_5S$	${}^4_6S$
x_1	163		489	816		
ops	653•41	653•163	653•367	653•1021	653•	653•

Table 33. How the 653-series might fit into a family.

Unfortunately, 653•163 cannot start a factor pair series since it is not itself a factor pair! In other words, 653•163 (= 106,439) is not an output of $x^2 + x + 41$.

Well, suppose that we revise the layout and fill in the various attributes as in Chart 4.

653-series Four, Five, and Six

What might the fourth, fifth, and sixth starting factor pairs be? What would these series' other attributes be?

Noting that the x_1:41 coefficients for the three known series are 4, 12, and 20, and that the $op2_1$:41 coefficients are 1, 9, and 25, then z is 1, 3, and 5 for these three series, respectively (1, 9, 25 = 1^2, 3^2, 5^2 and 4, 12, 20 = 4•1, 4•3, 4•5). Taking an educated guess that z for a fourth series in the sequence might be 7, the x_1:41 coefficient would be 4•7 = 28, and the $op2_1$:41 coefficient would be 7^2 = 49.

The three known x_1:41's are 4•41-1, 12•41-3, and 20•41-4. What would the fourth be? 28•41 minus what? On the second try, 28•41-6 = 1142 yielded a valid output of $x^2 + x + 41$ that is divisible by 653: 1,305,347 = 653•1999. This $op2_1$ value, 1999, yields an $op2_1$:41 value of 49•41-10.

Trying 9 for z for the fifth series, the x_1:41 value would be (4•9)•41-? = 36•41-?. After several attempts, 36•41-7 = 1469 yielded an output of $x^2 + x + 41$ divisible by 653: f(1469) = 2,159,471 = 653•3307.

By now the pattern of x_1:41's was plain. What gets subtracted from 4z•41 varies by 2, then 1, then 2, then 1, and so on. The sixth series' x_1:41 could be derived immediately as 44•41-9 = 1795.

Chart 5 contains the finished columns for 653-series four, five, and six. Once all of these values are filled in, their patterns are obvious. Before they are all filled in, however, these values are far from obvious, let alone their patterns.

Initial Extrapolation Difficulty

To illustrate the filling-in process, take the fourth series. What are its x spacing and op2 spacing values, given those of the first three series? All that we know to start with is that the sum of the x spacing coefficients equals the x_1:41 coefficient, 28. Similarly, all that we know to start with is that the sum of the op2 spacing coefficients equals the $op2_1$:41 coefficient, 49. Trying to extrapolate across to the fourth column for any of the individual spacing coefficients based on their sequences in the first three columns is too tricky. For example, for the odd x spacing coefficients the sequence goes 1, 7, 9, ?.

Guided Trial and Error

The only solution to finding these values is guided trial and error. We know that series four starts 653•1999. We also know that the op1 spacing for series four is 8•, 8•, since that is what it is for all 653-series. Therefore, x_2 = 661•?. By trying 1999+1, 1999+2, etc. for op2, we eventually succeed in finding a result that is an output of $x^2 + x + 41$ when op2 is 2027. So, x_2 = 661•2027. Similarly, since x_3 = 677•?, we keep going until we find x_3 = 677•2069. And so on. In this way we discover both the op2 spacing coefficients and the x spacing coefficients.

Cross-column Patterns

Do we have to do the same thing for each succeeding series? Only until we have enough data in our sequences across columns to discern the cross-column patterns. For example, given column four, our x spacing odd coefficients are now 1, 7, 9, 15, ?, ?. These values go up by 6, then by 2, then by 6 again. If it turns out that the value for column five goes up from that of column four by 2, then it is likely that we have our pattern. As one can see from Chart 5, the odd spacing coefficient for series five is 17.

Even if we don't yet see the pattern for the even x spacing coefficients, for example, we can deduce them given the pattern for the odd coefficients and the knowledge that the sum of the odd and even coefficients for a series equals the x_1:41 coefficient for that series.

The 1021-family

Rapid Projection of x_1:41

Armed with the confidence of having deduced the 653-family of factor pair series, the next venture, the prospective conquest of a family for 1021-based series, went swiftly.

The first three series in a 1021-family would start 1021•41, 1021•653 (a reversal of 653•1021), and 1021•1471. The x_1:41's for series four, five, and six could be projected thanks to the precedents set for coefficient and "subtrahend" spacings in the first three four families as follows:

Series	x_1:41	Coef.	÷v	Coef.÷v Spacing	Subtrahend	Subtrahend Spacing
${}^5_1S$	5•41- 1	5	1		1	
				3		3
${}^5_2S$	20•41- 4	20	4		4	
				2		1
${}^5_3S$	30•41- 5	30	6		5	

Table 34. Known 1021-series x_1:41 values.

Series	x_1:41	Coef.	÷v	Coef.÷v Spacing	Subtrahend	Subtrahend Spacing
^5_1S	5•41- 1	5	1		1	
				3		3
^5_2S	20•41- 4	20	4		4	
				2		1
^5_3S	30•41- 5	30	6		5	
				3		3
^5_4S	45•41- 8	45	9		8	
				2		1
^5_5S	55•41- 9	55	11		9	
				3		3
^5_6S	70•41-12	70	14		12	

Table 35. Known plus projected 1021-series x_1:41 values.

The x_1's Pan Out, But...

The resulting predicted x_1's for series four through six, 1837, 2246, and 2858, could be easily tested for validity. If the output of $x^2 + x + 41$ for each is divisible by 1021, then these are very probably series-starting x's. f(1837) = 3,376,447 = 1021•3307. f(2246) = 5,046,803 = 1021•4943. f(2858) = 8,171,063 = 1021•8003.

...Are They Really Series Seeds?

These results are encouraging, but further evidence is required before one can conclude that these are series-starting factor pairs. Subsequent factor pairs must stem from these "seed" pairs. How can we tell what factor pairs might follow from each prospective "starter pair"?

Beyond Trial and Error

We know what the op1 spacing must be for any 1021-based series, 10•, 15•. So, we know what the op1 operands for each following factor pair must be. Given these op1 values, we could use trial and error to find successful op2 operands for each succeeding op1 value that together yield an output of $x^2 + x + 41$. In other words, for series four, the next op1 is 1031, so we could hunt one-at-a-time for appropriate op2's until we hit 3343, which with 1031 yields f(1856). But, we don't have to use trial and error.

Projected op2 Spacing

Using similar analogizing as we did to deduce the x_1:41 values, we can deduce the op2 spacing values for series four through six:

Series	op2 Spacing	Odd Coef.	Odd Coef. Spacing	Even Coef.	Even Coef. Spacing
${}^5{}_1S$	${}^0{}_1$	0		1	
			8•1		7•1
${}^5{}_2S$	${}^8{}_8$	8		8	
			2•2		8•2
${}^5{}_3S$	${}^{12}{}_{24}$	12		24	

Table 36. Known 1021-family op2 spacing values.

Series	op2 Spacing	Odd Coef.	Odd Coef. Spacing	Even Coef.	Even Coef. Spacing
${}^5{}_1S$	${}^0{}_1$	0		1	
			8•1		7•1
${}^5{}_2S$	${}^8{}_8$	8		8	
			2•2		8•2
${}^5{}_3S$	${}^{12}{}_{24}$	12		24	
			8•3		7•3
${}^5{}_4S$	${}^{36}{}_{45}$	36		45	
			2•4		8•4
${}^5{}_5S$	${}^{44}{}_{77}$	44		77	
			8•5		7•5
${}^5{}_6S$	${}^{84}{}_{112}$	84		112	

Table 37. Known plus projected 1021-family op2 spacing values.

Verified Predictions

Do these predicted op2 spacings fit reality? For series four, given starting pair 1021•3307, the next pair would be 1031 (by necessity) • 3343 (by prediction). 1031•3343 is indeed an output of $x^2 + x + 41$. The next projected pair is 1061•3433. It, too, meets the test. Similarly, the projected pairs for series five and six all succeed. Thus, once again deduction has come through. Another reassuring point is that the predicted coefficients for op2 spacing in each series sum to a perfect square, and those perfect squares are indeed z^2, as appropriate. The filled in chart for the 1021-family appears as Chart 6.

Superfamily

It is now clear that we could go on and on like this, generating new families of series at will. Is this the end? Have we no more worlds to conquer? Hah! We have only scratched the surface.

Planes of Pairs

Cross-family Patterns

Given a pretty good sample of families of factor pair series, we might now be able to see cross-family patterns.

^{1}S's op1 Spacing

Just as we once changed series $^{1}_{1}S$'s op2 spacing from 0•, 1• to 2•, -1• due to pressure from the rest of the ^{1}S series' op2 spacings, we now see family ^{1}S's op<u>1</u> spacing is 0•, 1•, while the rest of the *families'* op1 spacings go 4•, 0•; 6•, 3•; 8•, 8•. This is the identical situation that made us change $^{1}_{1}S$'s op2 spacing from 0•, 1• to 2•, -1•. Should we do the same thing here?

$^{1}_{1}S$'s op1 Spacing vs. op2 Spacing

Before answering this question, let us examine the issues very carefully. We could change family ^{1}S's op1 spacing to 2•, -1• to fit the other families' op1 spacing sequence. ^{1}S's op1 spacing, if changed to 2•, -1•, would force $^{1}_{1}S$'s op2 spacing back to 0•, 1•, since they cannot both be 2•, -1•. Nor can both be 0•, 1•. One must be 2•, -1• when the other is 0•, 1•. To illustrate, if both were 2•, -1• we would get:

		41•41		
	2•1		2•1	
		43•43		
-1•2				-1•2
		41•41		
	2•3		2•3	
		47•47		
-1•4				-1•4
		43•43		
	2•5		2•5	
		53•53		

Table 38. The op1 and op2 spacings cannot both be 2•, -1•.

This is invalid, since these pairs do not yield outputs of $x^2 + x + 41$. Likewise, if both spacings are 0•, 1•, the pairs are invalid:

		41•41		
	0•1		0•1	
		41•41		
1•2				1•2
		43•43		
	0•3		0•3	
		43•43		
1•4				1•4
		47•47		
	0•5		0•5	
		47•47		

Table 39. The op1 and op2 spacings cannot both be 0•, 1•.

An Alternative op2 Spacing Sequence for ${}^1S$

Changing ${}^1_1S$'s op2 spacing to 0•, 1• is not the dead-end that it at first might seem to be. There *is* a sequence starting with 0•, 1• that can supply satisfactory op2 spacings for the ${}^1S$ family of series:

0•	4•	12•	24•	40•	
1•	0•	-3•	-8•	-15•	etc.

These alternative op2 spacings yield the same factor pairs when coupled with an op1 spacing of 2•, -1• as do the more familiar ${}^1S$ op2 spacings when coupled with 0•, 1•. The difference is that the sequence of op2's now goes up and down for ${}^1_3S$ and above. Only ${}^1_1S$ and ${}^1_2S$ have non-up-and-down op2's. With the original op2 spacings, only ${}^1_1S$ has up-and-down op2's; the rest of the series' op2's are strictly increasing (or at least non-up-and-down, as with ${}^1_2S$):

2•	4•	6•	8•	10•	12•
-1•	0•	3•	8•	15•	24•
41•41	41•163	41•367	41•653	41•1021	41•1471
0•1 2•1	4•1	6•1	8•1	10•1	12•1
41•43	41•167	41•373	41•661	41•1031	41•1483
1•2 -1•2	0•2	3•2	8•2	15•2	24•2
43•41	43•167	43•379	43•677	43•1061	43•1531
0•3 2•3	4•3	6•3	8•3	10•3	12•3
43•47	43•179	43•397	43•701	43•1091	43•1567
1•4 -1•4	0•4	3•4	8•4	15•4	24•4
47•43	47•179	47•409	47•733	47•1151	47•1663
0•5 2•5	4•5	6•5	8•5	10•5	12•5
47•53	47•199	47•439	47•773	47•1201	47•1723
1•6 -1•6	0•6	3•6	8•6	15•6	24•6
53•47	53•199	53•457	53•821	53•1291	53•1867

Table 40. ^{1}S with original op1 and op2 spacing (op1 spacing = 0•, 1•).

0•	4•	12•	24•	40•	60•
1•	0•	-3•	-8•	-15•	-24•
41•41	41•163	41•367	41•653	41•1021	41•1471
2•1 0•1	4•1	12•1	24•1	40•1	60•1
43•41	43•167	43•379	43•677	43•1061	43•1531
-1•2 1•2	0•2	-3•2	-8•2	-15•2	-24•2
41•43	41•167	41•373	41•661	41•1031	41•1483
2•3 0•3	4•3	12•3	24•3	40•3	60•3
47•43	47•179	47•409	47•733	47•1151	47•1663
-1•4 1•4	0•4	-3•4	-8•4	-15•4	-24•4
43•47	43•179	43•397	43•701	43•1091	43•1567
2•5 0•5	4•5	12•5	24•5	40•5	60•5
53•47	53•199	53•457	53•821	53•1291	53•1867
-1•6 1•6	0•6	-3•6	-8•6	-15•6	-24•6
47•53	47•199	47•439	47•773	47•1201	47•1723

Table 41. ^{1}S with alternative op1 and op2 spacing (op1 spacing = 2•, -1•).

Family or Superfamily: Which Should Switch?

Why didn't we switch to this sequence earlier, when we saw that 0•, 1• didn't fit in with the rest of the ^{1}S op2 spacings? Because it was much easier to change one exception to fit the rule than to change all cases except the exception to fit the exception.

Would we reconsider that decision now? Very possibly, in the light of what we are about to acknowledge. To recap, if we change ^{1}S's op1 spacing to 2•, -1• to fit the cross-family op1 spacing sequence, then we must change $^1_1$S's op2 spacing to 0•, 1• to work with its op1 spacing of 2•, -1•. This change of $^1_1$S's op2 spacing basically forces, by domino effect, the changing of the rest of the ^{1}S family's op2 spacings to that new sequence that starts with 0•, 1•. The net effect of this change is that the op2's of all ^{1}S series, from $^{1}_{3}S$ on, go up and down (an admittedly less "natural" way to list factor pairs than the original way).

The other alternative to changing ^1S's op1 spacing to 2•, -1• is to leave all of ^1S's op2 spacings unchanged and to change all of the other *families'* op1 spacings, via domino effect, to the same new sequence that starts with 0•, 1•. This would make all series within all *families* from ^3S on up have op1 spacings that go up and down. The up-and-down effect in this latter case is *much* more severe, applying not just within one family but across all but two families. The choice is clear-cut. The lesser of the messier situations is preferable: change ^1S's op2 spacing to 0•, 1• and ripple that change up through the rest of the ^1S family's series.

Messy Perfection?

Can the perfection of nature be so messy? This "messiness" is merely inconvenience, not disorder. We are confronted by, or if one prefers, treated to, an example of two equally valid alternative interpretations of one reality. This is reminiscent of the duality of the electron, "simultaneously" wave and particle, though the key is not so much that it is simultaneously both but that it is one or the other depending on context and therefore possibly "actually" neither per se, possibly something with at least those two faces for those two contexts yet with possibly other faces still.

We are inconvenienced by having two alternative descriptions and by having up-and-down number sequences soiling our otherwise normal number scenery.

Remember when we said that alternative numberings would come up again? Well, they have. And so they shall yet again.

The First Family Revised and Revisited

The impact of these changes to the ^1S family reaches to the x_1 spacings as well. The alternative chart for the ^1S family, where the op1 spacing is 2•, -1• appears in Chart 7. Notice that there *is* no impact on the x_1's, the sums of the x spacings, the sums of the op2 spacings, nor the $op2_1$:41 or x_1:41 values.

x_1:41 and Further Revision

When we list the values of the x_1:41 coefficients for the known series, for the known families, not only is there an interesting pattern -- some new knowledge is gained which changes our understanding of the 41- and 163-families. The x_1:41 coefficient for a series, as a reminder, is equivalent to the sum of the x spacing coefficients for that series, and both are equal to v•z. Remember that the x_1:41 coefficients for each series were all multiples of the same number, the family number, and that z was the result of dividing each such coefficient by that common factor, v.

x_1:41 Coefficients

Table 42 gives the x_1:41 coefficients for the first five series for each of the five known series families:

Family (v) →	1	2	3	4	5
Series (z)					
1	1	2	3	4	5
2	2	6	6	12	20
3	3	10	12	20	30
4	4	14	15	28	45
5	5	18	21	36	55

Table 42. The x_1:41 coefficients for the five known families.

Coefficient Spacings

Table 43 expands on Table 42 by including the spacings between the listed values:

v -->	1		2		3		4		5	
	1		2		3		4		5	
		1		4		3		8		15
	2		6		6		12		20	
		1		4		6		8		10
	3		10		12		20		30	
		1		4		3		8		15
	4		14		15		28		45	
		1		4		6		8		10
	5		18		21		36		55	

Table 43. x_1:41 coefficient spacing for the five known families.

Rewrite Inspired by op1 Spacings

One way to look at these spacings is to consider those of families one, two, and four nice and constant while those of families three and five are a little messy, alternating between two values. The trained eye, however, will quite possibly notice that the two alternating values for families three and five remind one of some other paired values. The alternating values here, 3 and 6, and 15 and 10, remind one of the op1 spacing coefficients for families three and five, 6•, 3• and 10•, 15•. Add to that the fact that family four's pair of eights can also be seen as the "reverse" of family four's op1 spacing coefficients, 8•, 8•, and a pattern starts to emerge.

If this is more than coincidence, then how would families one and two fit in? Their op1 spacing coefficients are 2•, -1• and 4•, 0• respectively. The coefficients sums for families one and two are 1 and 4. These are the spacings for families one and two in Table 43. So, let us rewrite Table 43 with 2, -1 and 4, 0 in place of 1 and 4, respectively:

v -->	1		2		3		4		5	
	1		2		3		4		5	
		2,-1		4,0		3		8		15
	2		6		6		12		20	
		2,-1		4,0		6		8		10
	3		10		12		20		30	
		2,-1		4,0		3		8		15
	4		14		15		28		45	
		2,-1		4,0		6		8		10
	5		18		21		36		55	
op1	2•		4•		6•		8•		10•	
spacing	-1•		0•		3•		8•		15•	

Table 44. Table 43 with (2,-1) and (4,0) in place of 1 and 4.

Reversed Pairs Come Closer

Does this help any? Well, we have got "funny" spacings with value pairs instead of single values for families one and two. Since the alternating values for the spacing for families three through five are in reverse order from the opl spacing coefficients, we should probably swap the two values in each spacing for families one and two as follows:

v -->	1		2		3		4		5	
	1		2		3		4		5	
		-1,2		0,4		3		8		15
	2		6		6		12		20	
		-1,2		0,4		6		8		10
	3		10		12		20		30	
		-1,2		0,4		3		8		15
	4		14		15		28		45	
		-1,2		0,4		6		8		10
	5		18		21		36		55	

opl	2•	4•	6•	8•	10•
spacing	-1•	0•	3•	8•	15•

Table 45. Table 43 with (-1,2) and (0,4) in place of 1 and 4.

Further Reconciliation

Is there any way to further reconcile families one and two with the other families? What if we get a little risqué and spread the (-1,2) and (0,4) pairs out as two single spacings each, as in:

-1	0	3	8	15
2	4	6	8	10
-1	0	3	8	15
2	4	6	8	10

Table 46. Spreading (-1,2) and (0,4) out into two values each.

Restructured Coefficient Sequences

What sense could these spacings possibly have for families one and two? The values separated by these spacings would have to change. Instead of 1, 2, 3, 4, 5, etc., family one's x_1 coefficient values would become:

1	
	-1
0	
	2
2	
	-1
1	
	2
3	
	-1
2	

Table 47. Family one's revised x_1 coefficient values.

Family two's values would become:

2	
	0
2	
	4
6	
	0
6	
	4
10	
	0
10	

Table 47. Family two's revised x_1 coefficient values.

These at first outlandish sequences can be seen at second glance to contain all of the values of the original versions of the sequences, only in every second position now. In other words, the first, third, fifth, and so on numbers in family one's new sequence are 1, 2, 3, and so on.

Implications of Revised Sequences

What are the implications of using such sequences in the charts for families one and two? Can it be done with results still meaningful? Yes. The number of series in both families doubles, or more accurately, an unexpected new series gets inserted between each of the previously accepted series.

Inserted Series and the New First Family

The "new" family one is analyzed in Chart 8, while the "new" family two is detailed in Chart 9.

Are There New Factor Pairs?

With these new "inserted" series, are there any new factor pairs to be found?

The first new series in family one, the series that starts with the factor pair 41•1, is the only series with any factor pairs not actually seen in previously known series. In other words, the rest of the inserted series in family one generate factor pairs already known of from the already known series in family one. For example, the new series that starts 41•43 generates:

		41•43		
	2•1		4•1	
		43•47		
-1•2				-3•2
		41•41		
	2•3		4•3	
		47•53		
-1•4				-3•4
		43•41		
	2•5		4•5	
		53•61		
-1•6				-3•6
		47•43		

Table 48. The new, inserted series that starts 41•43.

These factor pairs are the same as those generated by the series beginning with 41•41, only rearranged in sequence:

```
                41•41
        0•1             2•1
                41•43
    1•2                     -1•2
                43•41
        0•3             2•3
                43•47
    1•4                     -1•4
                47•43
        0•5             2•5
                47•53
    1•6                     -1•6
                53•47
                 etc.
```

Table 49. The factor pairs in the series beginning with 41•41.

This same effect of "nothing really new" occurs with the other inserted series in family one. The inserted series in family two are even obviously not going to contribute any new factor pairs -- these series are exact copies of their lefthand neighbors.

The First Inserted Series

So, only the first inserted series in family one yields previously unseen factor pairs. Just what are these factor pairs? Well, since the op2 spacing for the series is 0•, 0•, every op2 in the series is 1! The op1's are none other than $x^2 + x + 41$, from $x = 0$ on up, albeit via a stutter-step:

```
                41•1
          2•1
                43•1
    -1•2
                41•1
          2•3
                47•1
    -1•4
                43•1
          2•5
                53•1
                etc.
```

Table 50. The factor pairs in the series starting with 41•1.

These factor pairs need to be seen in a special light. The original search for factor pairs was in the context of finding the factors of *composite* outputs of $x^2 + x + 41$. Having 1 as a "factor" yields not only all of the composite outputs of $x^2 + x + 41$ but all of the prime outputs as well! So, in this sense, these special factor pairs are "uninteresting," yielding "trivial" factoring information.

Nevertheless, going through the exercise of expanding families one and two to conform to the format of the other families with respect to x_1:41 coefficient spacings yields a consistency across all of the families that in turn provides a certain security to build upon. Not only are all of the x_1:41 values now aligned, but the z's and other parameters are also now in step across the known families.

Other Family Charts

Charts similar to those of the first five families were now prepared for a few more families. This was straightforward, given the clarity of the patterns across families.

The First Superfamily

Families as Columns, Series as Points

It was now natural to summarize the families in one table representing each family in a column, with each series in a family represented only by its starting factor pair. This table in effect gives a glimpse of a superfamily:

w	^1S	^2S	^3S	^4S	^5S	^6S
1	41• 41	163• 41	367• 41	653• 41	1021• 41	1471• 41
2	41• 1	163• 41	367• 163	653• 367	1021• 653	1471• 1021
3	41•163	163• 367	367• 653	653•1021	1021•1471	1471• 2003
4	41• 43	163• 367	367•1019	653•1999	1021•3307	1471• 4943
5	41•367	163•1019	367•1999	653•3307	1021•4943	1471• 6907
6	41•167	163•1019	367•2609	653•4937	1021•8003	1471•11807

Table 51. The family of families of series of factor pairs.

Thorough Cross-family Pattern Analysis

Looking for patterns here, we see that the op1's are all familiar: 41, 163, 367, 653, 1021, etc. So, the items of interest here, if any, will involve the op2's.

Two Series-starting Factor Pair Patterns

Two patterns become evident. First, a factor pair in row 3 appears again reversed in row 2 of the column immediately to the right. For example, row 3 of column ^1S contains 41•163 while row 2 of column ^2S contains 163•41. Row 3 of column ^2S contains 163•367 while row 2 of column ^3S contains 367•163. And so on.

Second, the right-hand factor (op2) in row 5 of each column is the same as the op2 in row 4 of the column to the right. For example, op2 in row 5 of column ^1S is 367 while op2 in row 4 of column ^2S is also 367. (Technically, these op2's are $op2_1$'s, the *first* op2's in a series in each case.)

Predicting Starting Pairs

Formulas vs. Insight

Suppose that we wanted to predict the factor pairs in some column of the superfamily table. We could derive a formula for each individual family. We could then try to come up with a more general formula that works for all families. We might also note, however, that given the first three factor pairs in a column here the rest of the column can be generated. And, we can predict what the first three factor pairs in a column are.

The First Three Entries Per Column

Generating the Rest of the Column

First, how can we generate the rest of a column from its first three entries?

Constant $op1_1$

We know that the op1 for all entries in a column is constant. For example, for all entries in column ^3S the op1 is 367.

$op2_1$ Spacing

It just so happens that the $op2_1$'s in a column follow our by now familiar odd-and-even spacing theme:

entry 1		
	+ odd spacing coefficient • 1	
entry 2		
		+ even spacing coefficient • 2
entry 3		
	+ odd spacing coefficient • 3	
entry 4		
		+ even spacing coefficient • 4
entry 5		
etc.		

Table 52. $op2_1$ spacing for a column: odd and even spacing again.

To illustrate, take column ^4S. Its factor pairs' op2's are:

41			
	= 326 =	326•1	
367			
	= 654 =		327•2
1021			
	= 978 =	326•3	
1999			
	= 1308 =		327•4
3307			
etc.			

Table 53. Factor pair op2's for column ^4S.

Predicting the First Three Entries

$op1_1$'s

How can we predict the first three entries in a column? We know that their $op1_1$'s are a particular odd $op2_1$ in family one (^1S):

$$ {}^{S}op1_1 = {}^{1}_{2S-1}op2_1 $$

For example, ${}^{4}op1_1 = {}^{1}_{7}op2_1 = 653$.

The First Three $op2_1$'s

What are the first three $op2_1$'s, then, in a column? The first is always 41. The second is the odd ${}^{1}S$ $op2_1$ *before* this column's $op1_1$. The third $op2_1$ is the odd ${}^{1}S$ $op2_1$ *after* this column's $op1_1$ in ${}^{1}S$. For example, take column ${}^{5}S$. Its $op1_1$ is the 9^{th} $op2_1$ in ${}^{1}S$, its first $op2_1$ is 41, its second $op2_1$ is the 7^{th} $op2_1$ in ${}^{1}S$, and its third $op2_1$ is the 11^{th} $op2_1$ in ${}^{1}S$.

An Example

Let us work through an example. Let us generate the first several entries in family ${}^{7}S$ (column ${}^{7}S$). ${}^{7}op1_1$ will be $op2_1$ of ${}^{1}_{2\cdot 7-1}S = {}^{1}_{13}S$. We need a formula for $op2_1$ of ${}^{1}_{w}S$ where w is odd. The data for generating such a formula are:

w	${}^{1}_{w}S_1$	$op2_1$	$op2_1$:41
1	41• 41	41	1•41−0
2	41• 1	1	0•41+1
3	41•163	163	4•41−1
4	41• 43	43	1•41+2
5	41•367	367	9•41−2
6	41•167	167	4•41+3
7	41•653	653	16•41−3

Table 54. Data to help in generating a formula for $op2_1$ for ${}^{1}_{w}S$.

We can see that $op2_1$ of ${}^{1}_{w}S$ where w is odd is:

$$ [(w+1)2 \div 2]^2 \cdot 41 - (w-1) \div 2 $$

To verify this, if w = 7, this yields 653.

So, $op1_1$ of ${}^{7}S = {}^{1}_{13}S = 2003$. $op2_1$ of ${}^{7}_{1}S = 41$, $op2_1$ of ${}^{7}_{2}S = {}^{1}_{11}S = 1471$, and $op2_1$ of ${}^{7}_{3}S = {}^{1}_{15}S = 2617$. So, the first three entries in column 7 are:

2003· 41
2003·1471
2003·2617

Table 55. The first three entries in column 7.

The op1 Spacings

We can now derive the $op2_1$ spacings: odd spacing = 1471 - 41 = 1430, even spacing = (2617 - 1471) ÷ 2 = 573. The column therefore begins:

2003· 41		
	1430·1	
2003· 1471		
		573·2
2003· 2617		
	1430·3	
2003· 6907		
		573·4
2003· 9199		
	1430·5	
2003·16349		
etc.		

Table 56. The beginning of column 7.

Please remember that these columns in the superfamily table are not series of factor pairs, but series of *starting* factor pairs for series of factor pairs. In other words, each factor pair listed here represents a whole series of factor pairs. It is the first factor pair in the series that it represents. So, 2003·41 represents a whole series of factor pairs, 2003·1471 represents another series of factor pairs, and so on.

Are "Impure" $op2_1$'s Seeds, Too?

A natural question to ask when looking at Table 51 is "Are the $op2_1$'s other than those in the first three rows capable of being family "seeds," as the $op2_1$'s in rows 1-3 have been?" For example, the $op2_1$'s in rows 1-3, familiar numbers such as 41, 163, 367, and so on, all appear as $op1_1$'s for a whole family. Thus they are "seeds." Can "impure" numbers that appear as $op2_1$'s in Table 51, such as 1019, 1999, 3307, and so on, also serve as $op1_1$'s? The familiar numbers all occur as $op2_1$'s in ^1S and satisfy the formula $n^2 \cdot 41 - (n-1)$. The "impure" numbers do not meet either qualification.

Why ask? Because if the familiar, "pure" numbers are the only family seeds, then we now know the extent of the families --each has one of the $n^2 \cdot 41 - (n-1)$ numbers as its op1. If these are not the only seeds, then there is much more to look into.

1019

Other "Pure" Partners

I decided to try to see if 1019 is a seed. 1019 does appear twice in series starters: 163•1019 and 367•1019. What other series-starting factor pairs might 1019 be in? We might first try "pure" numbers as partners for 1019 and if they do not pan out then try numbers slightly less than the "pures." For example, we could try 653•1019, 1021•1019, 1471•1019, 2003•1019, 2617•1019, and so on. These do not work. One has a feeling that these partners cannot work, since 1019 is bypassed in each of their sequences in Table 51.

Providing Guidance in the Search

Rather than rather blindly searching for success that we do not even know is there, it would be good if we tried to provide some guidance to the search. Are there any hints in what data that we do have that might help narrow the field of candidates?

A Start on a Chart

Well, starting a chart for a prospective 1019 family, we have two entries so far, a column for 1019•163 and one for 1019•367 (see Chart 10).

Candidate Common Divisor of x_1:41 Coefficients

Noting that the x_1:41 coefficients, 10 and 15, have 5 as a common denominator, raises the possibility that 5 might be the common divisor for the whole 1019-family, if such a family exists.

"Pure" Family Common Divisors

Reinforcing that possibility is the fact that the common divisor of the x_1:41 coefficients for the known, "pure" families is the square root of the coefficient of $op2_1$:41 as well as being the "family number," v. To illustrate:

Family Number (= v)	op1	op1:41	√coef. of op1:41
1	41	1•41-0	1
2	163	4•41-1	2
3	367	9•41-2	3
4	653	16•41-3	4
5	1021	25•41-4	5
6	1471	36•41-5	6

Table 57. Pure family number vs. x_1:41, $op2_1$:41 coefficients.

1019 = 25•41-6. It is therefore reasonable to suggest that v for a 1019 family is 5. Here v might at least mean common divisor of x_1:41 coefficients (and x spacing sums), if not "family number."

Strategy for Finding Suitable x_1's

So, getting back to 10 and 15 being divisible by 5, other 1019-family x_1:41 coefficients might well be divisible by 5 as well. Since these are x_1's that we are constraining to be approximately a multiple of 5 times 41, we can search by trying candidate x_1's.

We need to apply $x^2 + x + 41$ to these candidate x_1's and then try dividing the result by 1019. If the result is divisible by 1019, then we succeed, and we find out what the other factor in the factor pair is.

Testing x_1 Candidates

We can first try candidate x_1's near 20•41 = 820 because 20 is the next multiple of 5 after 15; i.e., we can try f(820), f(819), f(818), etc. None pan out. Next are candidate x_1's near 25•41 = 1025. None pan out. Ditto for candidate x_1's near 30•41 = 1230.

f(1426)

With candidate x_1's near 35•41 = 1435, however, we succeed. f(1426) = 2,034,943 = 1019•1997.

Does f(1426) Start a Series?

We Know 1019's Spacing

We still have to determine if this factor pair starts a whole series or is just an isolated pair. Fortunately, this task is converted from another blind, possibly fruitless search to a simple exercise by realization that we know 1019's spacing from the other two known 1019-series, 20•, 5•. So, we need only find 1997's spacing and we are all set.

1997's Spacing

Do we have any information on that? Yes. We know that z for the 1019•1997 series would be 7 since the x_1:41 coefficient is 35 and the common divisor of those coefficients is 5, and those coefficients equal v•z. We also know that the sum of the op2 spacing coefficients for a series is z^2. So the sum of op2 spacing coefficients for a sequence with 1997 as $op2_1$ should be $7^2 = 49$.

x Spacing Constraint

We also know that the sum of the coefficients for x spacing equals v•z = 35. So, even using trial and error, if we increment x from x_1 (1426) and do not get a successful factor pair by the time that we reach $x_1 + 35$, then we know that we can give up.

f(1455)

Using what we know, therefore, $f(x_1)$ = 1019•1997, and $f(x_2)$ would be 1039•? where $x_2 \leq x_1 + 34 = 1460$. Also, "?" $\leq$ 1997 + 48 = 2045. Using a pocket calculator, we can determine that success occurs on $x_2 = 1455$. $op2_2 = 2039$. So, the second factor pair = 1039•2039.

Deducing the Rest of the Series

Given this second pair and the known spacing coefficient-sums, we can deduce the rest of the series. Since the sum of the x spacing coefficients is 35 and the odd coefficient is 29 (1455 - 1426), the even coefficient must be 6. Similarly, since the op2 spacing coefficients sum to 49 and the odd coefficient is 42 (2039 - 1997), then the even coefficient must be 7. To illustrate:

		f(1426)			1019•1997		
	29•1			20•1		42•1	
		f(1455)			1039•2039		
6•2			5•2				7•2
		f(1467)			1049•2053		

Table 58. The series started by f(1426).

f(1467)

Is f(1467) indeed 1049•2053? f(1467) = 2,153,597. 1049•2053 = 2,153,597.

The Rest of the Family

Given three established 1019-series, we can now deduce the rest of the 1019-family. Based on experience with the other family charts, given the first three z's we can deduce the rest. Here they are 2, 3, and 7. Their spacings are 1 and 4. The next z would be spaced 1 and the one after that would be spaced 4. In other words, the 1019 z's would go: 2, 3, 7, 8, 12, etc. Similarly, the x spacing coefficients, the op2 spacing coefficients, the $op2_1$:41 coefficients and subtrahends, and the x_1:41 coefficients and subtrahends all follow regular patterns predictable given the first three series in the family. See Chart 11 for the 1019-family.

The Door is Open

Now that we have established that there is a 1019-family, based on an "impure" number, the door is open for whole new sets of families. What might some of these others be?

Fourth and Sixth $op2_1$'s

^3S's

The 1019-family's fourth $op2_1$ is 2609. Noting in Table 51 that 2609 is ^3S's sixth $op2_1$ and 1019 is ^3S's fourth $op2_1$, the question arises as to any other such pairings of fourth and sixth $op2_1$'s in one family's series-starter sequence as series-starting pairs in other families.

^1S's

^1S's fourth and sixth $op2_1$'s are 43 and 167. They can be seen to start a series, although it is not part of a previously recognized family. Values that work are: x spacing = 7•, -5•, op2 spacing = 12•, -8•, and op1 spacing = 4•, -3•.

^2S's

^2S's fourth and sixth $op2_1$'s, 367 and 1019, start ^3S's fourth series. And, 1019 and 2609, ^3S's fourth and sixth $op2_1$'s, start 1019's fourth series. Is the 43•167 pair perhaps the start of some family's fourth series? It would be a 43-family, we can safely say.

^4S's, ^5S's, and ^6S's

What about ^4S's fourth and sixth $op2_1$'s, 1999 and 4937? Do they even yield an output of $x^2 + x + 41$ when multiplied together? Yes. 1999•4937 = 9,869,063 = f(3141). What about ^5S's fourth and sixth $op2_1$'s? 3307•8003 = f(5144). ^6S's fourth and sixth $op2_1$'s are 4943 and 11807. 4943•11807 = f(7639).

Series-starters All

These other factor pairs actually do start series:

x		$^{4}{}_{4}op2 \cdot {}^{4}{}_{6}op2$
3141		1999•4937
	43•1	
3184		2027•5003
	34•2	
3252		2069•5113
x		$^{5}{}_{4}op2 \cdot {}^{5}{}_{6}op2$
5144		3307•8003
	55•1	
5199		3343•8087
	71•2	
5341		3433•8311
x		$^{6}{}_{4}op2 \cdot {}^{6}{}_{6}op2$
7639		4943•11807
	67•1	
7706		4987•11909
	120•2	
7946		5141•12283

Table 59. Series started by pairing the fourth and sixth $op2_1$'s.

Other $op2_1$ Pairings?

Fourth and Seventh or Eighth?

Would other pairings of Table 51 $op2_1$'s also yield valid outputs of $x^2 + x + 41$ and therefore be possible series-starters in as yet unknown families? Pairing fourth and seventh $op2_1$'s of a family's starting pairs does not succeed. Nor does pairing fourth and eighth $op2_1$'s.

Other "2-apart" $op2_1$'s?

What about other "2-apart" $op2_1$'s, such as the fifth and seventh? Yes, these all succeed. For example ^3S's fifth and seventh $op2_1$'s yield 1999•4079 = f(2855), and even ^3S's sixth and eighth $op2_1$'s yield 2609•4933 = f(3587). *All* 2-aparts work for ^3S (41•653, 163•1019, 653•1999, etc.). All 2-aparts also work for ^4S. In fact, all 2-aparts work, period!

2-aparts

What shall we make of this startling development? Let us depict this effect, to start off with. Picking a given Table 51 family column, for example the ^3S column, we are seeing the following series-starting factor pair generation effect:

^3S	$op2_1 \cdot op2_3$	$op2_2 \cdot op2_4$	$op2_3 \cdot op2_5$	$op2_4 \cdot op2_6$
367• 41	41•653	163•1019	653•1999	1019•2609
367• 163				
367• 653				
367•1019				
367•1999				
367•2609				
etc.				

Table 60. 2-aparts generating series-starting factor pairs.

Generating Columns from Column 1, Revisited

Reminded of our discussion of generating the rest of Table 51's columns from its first column's first, third, fifth, seventh, etc. $op2_1$'s, let us revisit that topic before returning here.

Even-numbered Column 1 $op2_1$'s

If all alternate (2-apart) $op2_1$'s in a superfamily column generate series-starting factor pairs, then the even-numbered $op2_1$'s in the ^1S column, bypassed in the earlier discussion of column generation from the ^1S column, need to be taken into account.

"Even" Columns Generated

By following the same procedure for generating columns as was used earlier, the following "even" columns would be generated:

1•41	43• 41	167• 41	373• 41
1•41	43• 1	167• 43	373• 167
1•43	43•167	167• 373	373• 661
1•43	43• 47	167• 379	373•1039
1•47	43•379	167•1039	373•2027
1•47	43•179	167•1049	373•2657

Table 61. Generated "even" columns.

Are These Pairs Outputs of $x^2 + x + 41$?

Are these pairs actually factor pairs, and if so, does each start off a whole series of factor pairs? The answer to both questions is yes.

As far as these pairs being factor pairs, i.e outputs of $x^2 + x + 41$, we have already seen the pairs in the first even column, as the pairs generated by ^1S's second starter-pair, 41•1. The pairs in the second, third, and fourth "even" columns in Table 61 are not so obviously valid factor pairs, but this can be quickly verified by simple calculation. For example, is 373•2027 an output of $x^2 + x + 41$?

Formula for Finding if a Pair is an Output of $x^2 + x + 41$

It should be mentioned that there is a simple formula that can determine if a pair of numbers, when multiplied, yields an output of $x^2 + x + 41$. If the two numbers are "a" and "b," then if $a \cdot b = x^2 + x + 41$ for a whole number x, then

$$\frac{(4ab - 163)^{\frac{1}{2}} - 1}{2} \quad \text{is whole (and equals x).}$$

For example, $$\frac{(4 \cdot 373 \cdot 2027 - 163)^{\frac{1}{2}} - 1}{2} = 869.$$

So, 373•2027 = f(869). To verify this result, simply plug 869 into $x^2 + x + 41$. This yields 756,071. 373•2027 = 756,071.

The Derivation of the Formula

The formula was derived using the general solution to quadratic equations:

if $y = ax^2 + bx + c$

and given the general quadratic solution for x when $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm (b^2 - 4ac)^{\frac{1}{2}}}{2a}$$

then $0 = ax^2 + bx + c - y$

and $$x = \frac{-b \pm (b^2 - 4a(c-y))^{\frac{1}{2}}}{2a}$$

$$= \frac{-1 \pm (1^2 - 4 \cdot 1(41-y))^{\frac{1}{2}}}{2 \cdot 1} \qquad \text{for } y = x^2 + x + 41$$

$$= \frac{-1 \pm (1 - 4(41-y))^{\frac{1}{2}}}{2}$$

$$= \frac{-1 \pm (1 - (164-4y))^{\frac{1}{2}}}{2}$$

$$= \frac{-1 \pm (1 - 164 + 4y)^{\frac{1}{2}}}{2}$$

$$= \frac{-1 \pm (4y - 163)^{\frac{1}{2}}}{2}$$

$$= \frac{\pm (4y - 163)^{\frac{1}{2}} - 1}{2}.$$

Since y here is m•n,

$$x = \frac{\pm\,(4mn - 163)^{\frac{1}{2}} - 1}{2}.$$

A Choice of Solutions

Noting that there are actually two solutions for x depending on the sign of $(4mn - 163)^{\frac{1}{2}}$, we will work with the positive version of $(4mn - 163)^{\frac{1}{2}}$. The negative solution gives an alternative x that also generates a particular output of $x^2 + x + 41$. For example, although x = 4 generates the output 61, x = -5 also generates 61. This is another example of the choice of interpretations that permeates this subject.

Are These Pairs Series-Starters?

Getting back to our discussion of "even" columns, we can see that the pairs generated to form these columns as shown in Table 61 are factor pairs producing valid outputs of $x^2 + x + 41$.

Familiar Series in Disguise

What about these factor pairs starting series? If we look carefully at some of these prospective series starters, for example 167•373 and 167•379, we will realize that these are pairs already seen in other series. The series that starts off 163•367 next contains 167•373 and then 167•379. By appropriately manipulating 167•373 or 167•379 we can show that they start the same series that 163•367 does, only with the pairs in the series rearranged:

163•367	167•373	167•379
167•373	163•367	163•367
167•379	179•397	179•409
179•397	167•379	167•373
179•409	199•439	199•457
199•439	179•409	179•397

Table 62. 167•373 and 167•379 start series that echo 163•367's.

Why Bother?

This same phenomenon applies to all of the factor pairs in the "even" columns -- they start series that we have already seen via the "odd" columns, just rearranged in sequence. Why bother delineating the even columns, then? For the same reason that we included the even-numbered factor pairs in ${}^1S$ to begin with, for completeness and consistency.

Revising the Superfamily

If we acknowledge these even columns being generated from the ${}^1S$ column as helping to form the list of superfamily columns, then we can return to the discussion of all alternate $op2_1$'s of a column generating series-starters and know that this ties into the ${}^1S$ column generating the rest of the Table 51 superfamily. All that we need to do first is to revise our superfamily to make room for the even columns. The insertion of even columns between existing columns, surely reminiscent of the insertion of new columns between existing columns of information in our family 1 and 2 charts (Charts 8 and 9), leads to the expanded superfamily portrait below:

41• 41	41• 41	1•41	163• 41	43• 41	367• 41	167• 41	653• 41
41• 1	41• 1	1•41	163• 41	43• 1	367• 163	167• 43	653• 367
41•163	41•163	1•43	163• 367	43•167	367• 653	167• 373	653•1021
41• 43	41• 43	1•43	163• 367	43• 47	367•1019	167• 379	653•1999
41•367	41•367	1•47	163•1019	43•379	367•1999	167•1039	653•3307
41•167	41•167	1•47	163•1019	43•179	367•2609	167•1049	653•4937
41•653	41•653	1•53	163•1997	43•677	367•4079	167•2039	653•6899
41•373	41•373	1•53	163•1997	43•397	367•4933	167•2053	653•9181

Table 63. Portrait of the expanded superfamily.

Renaming Original Columns

One consequence of this revision is that our names for the original families, such as ${}^1S$, ${}^2S$, ${}^3S$, and so on, are no longer appropriate. ${}^1S$ is still ${}^1S$, but ${}^2S$ is now the third column and family, so it should be renamed to ${}^3S$. Old ${}^3S$ is now ${}^5S$, old ${}^4S$ is now ${}^7S$, and so on. This is not that bad an adjustment. It can be chalked up to growing pains. We must keep up with the times.

Back to 2-aparts

Column Generation Analogous to That From ^1S

Now, returning to alternate $op2_1$'s in Table 51 columns generating series starters, can whole series-starter columns, not just single series starters, be generated in a fashion analogous to that in which ^1S generates the whole family of columns in Table 63? Yes.

Building Whole Families of Child Columns

If we treat any Table 51, or now Table 63, column as a "parent" column like the ^1S column can be, then, by using the same construction method, we can build a whole family of "child" columns.

The Construction Method

To review, the construction method is to populate the first three series-starting factor pairs in a child column. These three pairs are sufficient to generate the rest of a column.

The first pair in child column n consisted of the parent $op1_1$ as the child $op2_1$, and the n^{th} *odd* parent $op2_1$ as the child $op1_1$. With the new, expanded concept of the superfamily, this latter mapping becomes: the n^{th} parent $op2_1$ is the child $op1_1$. For example, previously the 163-column was column 2 and it took its $op1_1$ from the parent column's second odd $op2_1$, 163 (the third $op2_1$ overall). Similarly, the 367-column, formerly column 3, took its $op1_1$ from the parent column's third odd $op2_1$, 367 (the fifth $op2_1$ overall). Now the 163-column is column 3 and the 367-column is column 5. They just take their $op1_1$'s from the parent column's third and fifth $op2_1$'s, respectively (the n^{th} $op2_1$ instead of the n^{th} odd $op2_1$).

The second pair in child column n, in the old counting, had the $(n-1)^{st}$ parent $op2_1$ as its $op2_1$. Now it has the $(n-2)^{nd}$ $op2_1$ as its $op2_1$. Its $op1_1$ as well as the third pair's was, and remains, the same as the first pair's $op1_1$ ($op1_1$ is constant for a family's column).

The third pair's $op2_1$ was the $(n+1)^{st}$ parent $op2_1$. It is now the $(n+2)^{nd}$ parent $op2_1$.

To recap what the method is now:

For child column n series-starting factor pair 1:

$op1_1$ = parent-column series-starting factor pair n $op2_1$

$op2_1$ = parent column $op1_1$

For child column n series-starting factor pair 2:

$op1_1$ = parent-column series-starting factor pair n $op2_1$

$op2_1$ = parent-column series-starting factor pair n-2 $op2_1$

For child column n series-starting factor pair 3:

$op1_1$ = parent-column series-starting factor pair n $op2_1$

$op2_1$ = parent-column series-starting factor pair n+2 $op2_1$.

An Example

Planes

As an example, let us generate a superfamily table from the 367-column of the original superfamily. Another name for a superfamily, or endless list of endless column-lists, is a "plane" of series-starting factor pairs. From here on we shall use the term "plane" to mean superfamily. So, our original superfamily is "the first plane," while any "child" superfamilies, generated from columns in the first plane, are "child planes."

First Pairs First

Let us begin building the child plane based on first-plane column 5, the 367-column, by filling in the first series-starting factor pair in each child column:

Parent	1	2	3	4	5
367• 41	41• 367	163• 367	653• 367	1019• 367	1999• 367
367• 163					
367• 653					
367•1019					
367•1999					
367•2609					
367•4079					

Table 64. Generating the first series-starters in child plane columns.

Note the pattern of child $op1_1$'s. Also, note that all of the child $op2_1$'s are the parent $op1_1$.

Second Pairs

Now let us fill in the second series-starting factor pair in each child column:

Parent	1	2	3	4	5
367• 41	41• 367	163• 367	653• 367	1019• 367	1999• 367
367• 163	41• 163	163• 41	653• 41	1019• 163	1999• 653
367• 653					
367•1019					
367•1999					
367•2609					
367•4079					

Table 65. Generating the second series-starters in child plane columns.

Parent Pairs Numbers -1 and 0

Note that the $op1_1$'s remain constant within a column. Also, the child $op2_1$'s in child column n's second series-starting factor pair equal parent pair n-2's $op2_1$. Doesn't this mean that child column 1's second $op2_1$ comes from parent pair -1? And child column 2's second $op2_1$ from parent pair 0? Yes. These parent pairs are meaningful and do indeed have the correct values for our purposes. By backtracking from the first few values in the parent-pair series, we can see what values pair -1 and pair 0 have:

Parent Pair #	Parent Pair	op2 Spacing	
-1	367•163		
		122•-1	
0	367• 41		
			245•0
1	367• 41		
		122•1	
2	367•163		
			245•2
3	367•653		

Table 66. Backtracking to parent pairs -1 and 0.

Third Pairs

Continuing with our example, we now fill in the third series-starting factor pair in each child column:

Parent	1	2	3	4	5
367• 41	41• 367	163• 367	653• 367	1019• 367	1999• 367
367• 163	41• 163	163• 41	653• 41	1019• 163	1999• 653
367• 653	41• 653	163•1019	653•1999	1019•2609	1999•4079
367•1019					
367•1999					
367•2609					
367•4079					

Table 67. Generating the third series-starters in child plane columns.

Note that the third $op2_1$ in child column n comes from parent-pair $op2_1$ n+2.

Op2 Spacing

An Illustration

Given the first three series-starting factor pairs in a column, the op2 spacing for the column can be derived. We shall illustrate this for the first child column in our example:

Child Column 1	Op2 Spacing	
41•367		
	-204•1	
41•163		
		245•2
41•653		

Table 68. Deriving a column's op2 spacing from three entries.

The Rest of a Column

Given the op2 spacing for a column, the rest of the column's pairs can be calculated:

Child Column 1	Op2 Spacing
41• 367	
	-204•1
41• 163	
	245•2
41• 653	
	-204•3
41• 41	
	245•4
41•1021	
	-204•5
41• 1	
	245•6
41•1471	
	-204•7
41• 43	

etc.

Table 69. The rest of a column given the op2 spacings.

Generating Any Child Column

The same method can be used to derive the contents of the other child columns generated by first-plane column 5 (the 367-column).

Generating Any Child Plane

Furthermore, the same overall method used to generate the child plane from first-plane column 5 can be used to generate a child plane from any first-plane column. A filled-in portion of the plane generated from first-plane column 5 is shown below:

	-204• 245•	-326• 489•	-326• 979•	-204• 1223•	286• 1713•
Parent	1	2	3	4	5
367• 41	41• 367	163• 367	653• 367	1019• 367	1999• 367
367• 163	41• 163	163• 41	653• 41	1019• 163	1999• 653
367• 653	41• 653	163•1019	653• 1999	1019• 2609	1999• 4079
367•1019	41• 41	163• 41	653• 1021	1019• 1997	1999• 4937
367•1999	41•1021	163•1997	653• 4937	1019• 6889	1999•11789
367•2609	41• 1	163• 367	653• 3307	1019• 5869	1999•13219
367•4079	41•1471	163•3301	653• 9181	1019•13207	1999•23497
367•4933	41• 43	163•1019	653• 6899	1019•11779	1999•25499
367•6893	41•2003	163•4931	653•14731	1019•21563	1999•39203

Table 70. The plane generated from first-plane column 5.

Please note that the numbers at the top of each child column are the odd and even $op2_1$ spacings for the column, respectively.

Extending Our Notation to Include "Plane"

$^{u:v}{}_{w}S$

It is time to extend our notation to include plane in the identification of factor pair series. As we introduced the designation $^{v}{}_{w}S$ earlier to denote series w within family v, now we could introduce $^{u:v}{}_{w}S$ to denote series w in family v within plane u.

$^{1:u:v}{}_{w}S$

However, we must go one step further. Since all planes ultimately stem from the first plane, we denote series w in family v in plane u stemming from the first plane as $^{1:u:v}{}_{w}S$. Thus, for example, a family (column) in the plane generated from first-plane column 5 would be $^{1:5:v}S$. A factor pair series w within that family would be $^{1:5:v}{}_{w}S$. Please note that although the "S" in these designations stands for "series," when we are referring to a structure larger than a single series, for example a family of series or a family of families (a plane), the "S" can still be interpreted as "series." In the case of these larger structures, the term "series" simply has the plural meaning. So, for example, while we can read $^{1:u:v}{}_{w}S$ as the particular series w in family v in plane u, we can read $^{1:u:v}S$ as the series (plural) in family v in plane u.

$^{1:u:v}{}_{w}S_{x}$

An Illustration

A particular factor pair x within that factor pair series would be $^{1:5:v}{}_{w}S_{x}$. To illustrate, let us see how we would designate progressively more specific points in the plane generated from first-plane family/column 5, the 367-column (see Table 63).

The 367-column is $^{1:5}S$, the fifth column of series-starting factor pairs in plane 1. Column 4 in the child plane generated from the 367-column is $^{1:5:4}S$, the column with 1019 as its op1 (see column 4 in Table 70). The third factor pair series in that column, the one starting with 1019•2609, is designated as $^{1:5:4}{}_{3}S$. Finally, the second factor pair in that factor pair series, 1039•2657, is $^{1:5:4}{}_{3}S_{2}$.

Relative x vs. Absolute x

Please note that x here is the sequence number of a factor pair within its series, "relative" x, not the "absolute" x that we have used when discussing f(x). While the relative x in this example is 2, the absolute x for factor pair 1039•2657 is 1661, as 1039•2657 = 2,760,623 = f(1661).

Denoting the Plane as a Whole

Also, please note that to denote this whole child plane we would use ${}^{1:5:v}S$, not ${}^{1:5}S$, because the superscript to the immediate upper left of "S" always denotes a family/column. Any plane designation is appended to the left of the family/column designation. If we are referring to the columns in the first plane we write ${}^{1:v}S$. Column 5 in particular is ${}^{1:5}S$. When referring to the columns in a child plane of the first plane, we write ${}^{1:u:v}S$.

Grandchild Plane Designation

If we find that child plane columns can generate their own child planes, then we would designate such "grandchild" planes as ${}^{1:t:u:v}S$ and their series as ${}^{1:t:u:v}{}_{w}S$. And so on, down as many levels as desired.

Appending Any Number of 1's on the Left Side of the Superscript

One final note: Column 1 of the first plane can generate the whole first plane, including itself as the first child column. Working in reverse, if the first plane is ${}^{1:v}S$, then that special first column is ${}^{1}S$, just as we have labeled it all along. What is not obvious, though, is that the effect of appending any number of 1's to the plane-identifying superscript is the same no matter how many 1's are included. In other words, ${}^{1:v}S \equiv {}^{1:1:v}S \equiv {}^{1:1:1:v}S$ etc.

This holds because no matter how many leading 1's appear in the superscript, they always generate ${}^{1}S$, "the first family," down to the rightmost of the consecutive leading 1's. In other words, since ${}^{1}S$ = the first family and ${}^{1:1}S$ = the first family and ${}^{1:1:1}S$ = the first family and so on, any column or plane generated from one of those is the same when generated from any other of those. For example, the fifth column in the plane generated from ${}^{1}S$ (${}^{1:5}S$) is the same as the fifth column generated from ${}^{1:1}S$ (${}^{1:1:5}S$) and they are both the same as the fifth column generated from ${}^{1:1:1}S$ (${}^{1:1:1:5}S$), and so on. So, ${}^{1:5}S \equiv {}^{1:1:5}S \equiv {}^{1:1:1:5}S$ etc.

Some Questions

Are These Pairs All Really Series Starters?

Are the factor pairs generated by the plane construction method recently discussed all series starters? No proof has been attempted to date, but all cases tested have proven successful.

Can Child Planes Have Children of Their Own?

Are the child plane columns capable of generating children of their own, grandchild planes? That question was pursued soon after the discovery of plane generation. We shall see shortly, after a brief digression into the realm of z, how that question is answered.

Are Any Pairs Not Generated by the First Plane or Its Children?

A related, yet different question also arises: Are there any factor pairs *not* generated by the first plane or its children? Again, no proof of the exhaustiveness of the plane family has been pursued, yet no anomalous factor pairs are known.

Getting to Know the First Plane and Its Children

Calculating Attribute Values, and Ultimately Factor Pairs, Given Coordinates

With the discovery of child planes, a whole new world opened up for exploration. Before going on to the question of child planes themselves generating (begetting?) child planes of their own, we wanted to better familiarize ourselves with two levels of plane -- the first plane and its immediate offspring. After all, we had only recently expanded our horizons to encompass a whole plane, the first superfamily, or first plane. We might want to see if we could derive some formulas or rules so that we can calculate any op2 or x spacing or what-have-you, given a factor pair series's "coordinates" in the first plane or in a first-generation child plane. Ultimately it would be nice to be able to calculate a specific factor pair given only its designation, i.e. ${}^{1...v}{}_{w}S_{x}$.

A Short but Important Side-tour into One Corner of the Realm of z

Before we embark on even that journey, we will travel briefly into one corner of the realm of z. This side-jaunt is actually not that irrelevant to looking for general formulas for parameters like op2 and x spacing. z is another parameter and as it will turn out much later, a very important one. For now, we will investigate a small portion of the total territory of z-value occurrence.

The z-lists for the First Several Families

Looking back at the first plane, the lists of z's for the first several families reveal interrelationship after first appearing autonomous. After the initial introduction via insertion of the even-numbered families, the z-lists for the first several families were as follows:

w	^1s	^2s	^3s	^4s	^5s	^6s	^7s
1	1	1	1	1	1	1	1
2	0	1	1	0	2	1	3
3	2	1	3	2	4	3	5
4	1	1	3	1	5	3	7
5	3	1	5	3	7	5	9
6	2	1	5	2	8	5	11

Table 71. The z-lists for the first several families.

Derived in the Usual Way

These values for z for each factor pair series were derived in the usual way, by taking the square root of the $op2_1$:41's coefficient (or of the sum of the op2 spacing coefficients), or by dividing the constant common divisor for a chart into the x_1:41 coefficient (or into the sum of the x spacing coefficients) for a series.

Cross-list Analysis

The values of z for a given family/column/v are orderly and each sequence can be extrapolated. It is when we look for relationships across the sequences that we find something to be resolved.

A Sequence to the Sequences?

Yes, after a sequence of z's first appears, in an odd column, it recurs three columns later. For example, column 1's z sequence reappears as column 4's sequence as well. But, is there an underlying sequence to the sequences?

Glimpse of Underlying Patterning After Including Spacings

By enhancing Table 71 to include differences between z's, the underlying patterning can begin to be glimpsed:

w	^{1}S	^{2}S	^{3}S	^{4}S	^{5}S	^{6}S	^{7}S	^{8}S	^{9}S
1	1	1	1	1	1	1	1	1	1
	-1	0	0	-1	1	0	2	1	3
2	0	1	1	0	2	1	3	2	4
	2	0	2	2	2	2	2	2	2
3	2	1	3	2	4	3	5	4	6
	-1	0	0	-1	1	0	2	1	3
4	1	1	3	1	5	3	7	5	9
	2	0	2	2	2	2	2	2	2
5	3	1	5	3	7	5	9	7	11
	-1	0	0	-1	1	0	2	1	3
6	2	1	5	2	8	5	11	8	14

Table 72. Spacing between z's.

So far, so good. Each z sequence has an odd spacing amount and an even spacing amount. For example, for ^{1}S, the first, third, and fifth z spacings are -1, while the second and fourth z spacings are 2. Now, what cross-column pattern of spacings can we find, if any?

Cross-column Spacing Analysis

All But One Even Spacing = 2

Well, at first glance the odd spacings don't seem to follow any pattern. What about the even spacings? Bam! All of the even spacings equal 2 excepts for column 2's.

Is the Exception Valid?

Is this a valid exception and the seeming synchrony of the other columns merely coincidence? This is hardly likely in the orderly number world that we have been venturing in.

Tinkering With Column 2

Even z-Spacing Set to 2

What would happen if we interceded in the second column's sequencing and changed its even spacing to 2? This would yield the following sequence of z's given the same starting value of "1" and the same odd spacing of "0":

w	^2S's	z's
1	1	
		0
2	1	
		2
3	3	
		0
4	3	
		2
5	5	
		0
6	5	

Table 73. Column 2's z's with their even spacing changed to 2.

Valid But Irrelevant Result

This is a valid-looking sequence, matching ^3S's and ^6S's, but there is one major problem with it -- these values don't fit the chart data for ^2S at all!

A Relevant Spacing Sequence with Even Spacing = 2?

Is there any way of changing the even spacing to 2 while preserving meaningfulness of the changed sequence? As it turns out, there is. What shall guide us to that solution, or resolution, is looking at the odd spacings more closely.

Looking Closely at the Odd Spacings

As they stand, the odd spacings are:

-1
0
0
-1
1
0
2
1
3

Table 74. The odd spacings for z's looking across columns.

Their Spacings

By including *their* differences we get:

Odd Spacing	Difference
-1	
	1
0	
	0
0	
	-1
-1	
	2
1	
	-1
0	
	2
2	
	-1
1	
	2
3	

Table 75. Spacings between the odd spacings for z's.

Looking at these difference values, it becomes evident that after the first two they pair as -1, 2 repeatedly. This suggests that perhaps the first two should really be -1, 2 as well.

Setting the First Two to -1, 2

What is the effect of changing the first two to -1, 2? Working backward from where the -1, 2 differences originally begin yields:

Odd Spacing	Difference
-1	
	-1
-2	
	2
0	
	-1
-1	
	2
1	
	-1
0	
	2
2	
	-1
1	
	2
3	

Table 76. Spacings between odd z spacings with two revisions.

The Only Effect

Interesting. The only effect of this change towards consistency is that the second odd spacing changes to -2. Remembering where these odd spacings come from, the effect of the above change is that ^2S's odd spacing becomes -2.

Revised z Sequence

Coupling this change with the prior one where we changed ^2S's even spacing to 2 yields ^2S's z's as follows:

z	odd spacing	even spacing
1		
	-2	
-1		
		2
1		
	-2	
-1		
		2
1		
	-2	
-1		
		2
1		
	etc.	

Table 77. ^{2}S's z's revised.

If Only This Were Meaningful

If this z sequence for ^{2}S were only meaningful with respect to ^{2}S's chart and things like x spacing and $op2_1$:41 coefficients, then we would have a z sequence whose even and odd spacings fit smoothly with those of all of the other families.

How This z Sequence Could Be Valid

But how could this alternating sequence of 1's and -1's for z be meaningful? Well, for those parameters relevant to z where they relate to z^2, -1^2 is the same as 1^2. And, the other two parameters in the chart for ^{2}S that relate to v•z happen to be 0. So, this revised z sequence is consistent with the empirical data for family ^{2}S. It works. Perhaps it would be best at this point if we were to look at the rather unique chart for ^{2}S. Please see Chart 12 in Appendix A.

Cross-column Consistency and Deeper Insight

So, we have gained not only corrected and more closely consistent z sequences, but deeper insight into the workings of this world as well.

Charting the Child Planes

Charts for the First Nine Child Planes

In the spirit of the original documentation of the various parameters such as op2 spacing for the first families, in Charts 1 through 4, a systematic sample charting of such information was prepared for the families of the first nine child planes. This was not to skip the parent, first plane -- it is its own first child plane, so that it is covered indirectly in the survey of the children of the first plane. Charts 13-66 in Appendix A contain the data for those nine child planes, six families of series each, the first three series per family being given. Only the first three series per family are given because the rest of a family's data can be derived from just its first three series' parameter values.

Invaluable Data for Generalization and Verification

Having the data in all of these charts on hand is invaluable for later studies that require such raw data. Any attempt to derive formulas or generalizations about the parameters of series for a whole plane, or even a whole set of planes, needs such data as the charts provide, both as input to the formulation process and as test data for verifying the formulations arrived at.

Only Parameter Formula Results But ± Discussed in Depth

We are going to bypass the details of the derivation of the formulas for the various parameters for the first plane's child planes and simply present the results. We will nevertheless need to discuss the previously undeveloped parameter of "±" in some depth.

Just Listing the Formulas Requires Explanation

Even just listing the resulting formulas will take a certain amount of explanation, as they will partly be expressed in terms of intermediate constructs not previously introduced. In fact, it is best to begin by introducing those constructs.

Intermediate Constructs

A Happy Medium

Intermediate constructs are used here so that on the one hand the final forms of the formulas are not too cumbersome due to having too many terms to deal with, and so that on the other hand the final forms of the formulas are not too compact and abstract, and distant from what is easily grasped and remembered. Examples of intermediate constructs are the simple concepts A, B, C, and D discussed earlier.

Review of A, B, C, and D

To review A, B, C, and D, since they are the base items that the intermediate constructs and ultimately the formulas are built from:

$A_x \equiv \lfloor x \div 2 \rfloor$

$B_x \equiv \lceil x \div 2 \rceil - 1$

$C_x \equiv A_x^2$

$D_x \equiv B_x(B_x + 1)$

where $\lfloor \; \rfloor$ means "rounded down" and $\lceil \; \rceil$ means "rounded up".

A and B are useful for describing series with odd and even spacings of the form

odd spacing • 1
even spacing • 1
odd spacing • 1
even spacing • 1
etc.

C and D are useful for series with spacings of the form

odd spacing • 1
even spacing • 2
odd spacing • 3
even spacing • 4
etc.

Subscript notation such as "A_x" is used rather than parenthetic notation such as "A(x)," even though these constructs are really functions, since the subscript notation is more compact and thus helps keep the expressions easier to read and grasp.

The Full List of Constructs

The full list of intermediate constructs is:

$$A_x \equiv \lfloor x \div 2 \rfloor$$

$$B_x \equiv \lceil x \div 2 \rceil - 1$$

$$C_x \equiv A_x^2$$

$$D_x \equiv B_x(B_x + 1)$$

$$E_x \equiv A_x(A_x - 2)$$

$$F_x \equiv B_x(B_x - 1)$$

$$G_x \equiv 2(B_x - A_x) + 1$$

$$H_x \equiv 2(D_x - C_x) + 1$$

$$I_x \equiv 2(F_x - E_x) + 1$$

$$J_x \equiv 81(G_x + A_x) + A_x$$

$$K_x \equiv 40(H_x + C_x) + D_x + 1$$

$$L_x \equiv 40(I_x + E_x - 2) + F_x$$

$$M_{xy} \equiv A_xA_y + G_y$$

$$N_x \equiv \tfrac{1}{2}(3G_x - A_x)$$

Table 78. Intermediate constructs used in general calculations.

First-plane Child-plane Factor Pair Series General Formulas

The general formulas for the various factor pair series parameters in the first plane's child planes are:

$$^{1:u:v}op1_1 \equiv {}^{1:u}{}_{v}op2_1 \equiv J_uD_v + L_uC_v + 41$$

$$^{1:u:v}c_{od}op1 \equiv 2G_uD_v + (I_u - 1)C_v$$

$$^{1:u:v}c_{ev}op1 \equiv 2A_uD_v + (E_u - 1)C_v + 1$$

$$^{1:u:v}{}_{w}op2_1 \equiv [2L_uA_v + J_u(2B_v+1)]D_w + [L_uE_v + J_u(F_v-1) + 41]C_w + K_u$$

$$^{1:u:v}{}_{w}z \equiv (G_u + A_u)M_{vw} + G_vA_w$$

Parameters Addressed

The parameters addressed are:

$^{1:u:v}op1_1$

the op1 value starting off all factor pair series in family v in plane u

$^{1:u:v}c_{od}op1$

the coefficient for the odd spacing in the op1 sequence in all factor pair series in family v in plane u

$^{1:u:v}c_{ev}op1$

the even spacing coefficient for the op1 sequence in family v in plane u

$^{1:u:v}{}_{w}op2_1$

the op2 value starting off a particular factor pair series in family v in plane u

$^{1:u:v}{}_{w}z$

the value of z for factor pair series w in family v in plane u.

Parameters Not Addressed

Work on generalizing other parameters was done but will not be presented here.

Parallels Between Parameters

Before making these abstract formulas more understandable by working through some examples, we will point out some interesting parallels between some of the parameters.

A Family Example

We will reproduce and extend to five columns wide a portion of the chart for family $^{1:3:6}$S (Chart 30) to help illustrate further the intricate interrelationships permeating this number pattern realm -- please see Chart 67 in Appendix A.

Internal Spacings Included

Now we will show the same data but with internal spacings indicated for x spacing and op2 spacing coefficients, and for x_1:41 and $op2_1$:41 subtrahends -- please see Chart 68.

Parallel Spacing-spacings

x Spacing Coefficients vs. op2 Spacing Coefficients

Notice how the spacings between spacings parallel each other between the x spacing coefficients and the op2 spacing coefficients. While the odd x spacing coefficients ascend by 8 and then by 22, repeatedly, the odd op2 spacing coefficients for those same factor pair series ascend by 8•1, then 22•2, then 8•3, 22•4, and so on. The same parallelism exists for the even spacing coefficients' own spacings: -3, -2, -3, -2, etc. vs -3•1, -2•2, -3•3, -2•4, etc.

x_1:41 Subtrahends vs. $op2_1$:41 Subtrahends

Moving down to the subtrahends for x_1:41 vs. for $op2_1$:41, the same kind of parallelism occurs: -1, -5, -1, -5, etc. vs. -1•1, -5•2, -1•3, -5•4, etc. Please note that we are here considering subtrahends as increasing, the larger the amount that gets subtracted -- thus, the spacings between subtrahends are treated as positive numbers in Chart 68 though more and more is being subtracted as one moves across the chart to the right.

x Spacing Coefficient Sums vs. x_1:41 Coefficients, and op2 Spacing Coefficient Sums vs. $op2_1$:41 Coefficients

And, of course, we already know about the parallelism between the x spacing coefficient sums and the x_1:41 coefficients, and between the op2 spacing coefficient sums and the $op2_1$:41 coefficients.

Getting Back to General Formulas

Getting back to general formulas, the ultimate objective of such formulation would be the ability to predict what a given factor pair within a given factor pair series is.

op2 Spacing is Conspicuously Absent

No such formula has yet been presented. Further, an essential ingredient in such a calculation would be op2 spacing, which itself is conspicuously absent from the list of parameters with formulas so far discussed.

± Will Lead the Way

After our examples with the existing formulas we shall introduce the concept of ±, which ties directly into op2 spacing. From there the way is clear to a method for deriving a given factor pair.

Formula Usage Examples

Let us work through two examples of usage of the general formulas that we have so far. The parameters that we will derive will be the starting op1 for a factor pair series (and for the whole family), the op1 spacing coefficients, the first op2 for the series, and z for the series. The two example series will be ${}^{1:2:3}{}_{4}S$ and ${}^{1:9:6}{}_{3}S$.

Five ${}^{1:2:3}{}_4S$ Parameters

${}^{1:2:3}{}_4S$:

$$
\begin{aligned}
u &= 2\\
v &= 3\\
w &= 4\\
A_u &= \lfloor u\div 2\rfloor = \lfloor 2\div 2\rfloor = \lfloor 1\rfloor = 1\\
A_v &= \lfloor v\div 2\rfloor = \lfloor 3\div 2\rfloor = \lfloor 1.5\rfloor = 1\\
A_w &= \lfloor w\div 2\rfloor = \lfloor 4\div 2\rfloor = \lfloor 2\rfloor = 2\\
B_u &= \lceil u\div 2\rceil - 1 = \lceil 2\div 2\rceil - 1 = \lceil 1\rceil - 1 = 1 - 1 = 0\\
B_v &= \lceil v\div 2\rceil - 1 = \lceil 3\div 2\rceil - 1 = \lceil 1.5\rceil - 1 = 2 - 1 = 1\\
B_w &= \lceil w\div 2\rceil - 1 = \lceil 4\div 2\rceil - 1 = \lceil 2\rceil - 1 = 2 - 1 = 1\\
C_u &= A_v^{2} = 1^2 = 1\\
C_v &= A_v^{2} = 1^2 = 1\\
C_w &= A_w^{2} = 2^2 = 4\\
D_u &= B_u(B_u+1) = 0(0+1) = 0(1) = 0\\
D_v &= B_v(B_v+1) = 1(1+1) = 1(2) = 2\\
D_w &= B_w(B_w+1) = 1(1+1) = 1(2) = 2\\
E_u &= A_u(A_u-2) = 1(1-2) = 1(-1) = -1\\
E_v &= A_v(A_v-2) = 1(1-2) = 1(-1) = -1\\
F_u &= B_u(B_u-1) = 0(0-1) = 0(-1) = 0\\
F_v &= B_v(B_v-1) = 1(1-1) = 1(0) = 0\\
G_u &= 2(B_u-A_u) + 1 = 2(0-1) + 1 = 2(-1) + 1 = -2 + 1 = -1\\
G_v &= 2(B_v-A_v) + 1 = 2(1-1) + 1 = 2(0) + 1 = 0 + 1 = 1\\
G_w &= 2(B_w-A_w) + 1 = 2(1-2) + 1 = 2(-1) + 1 = -2 + 1 = -1\\
H_u &= 2(D_u-C_u) + 1 = 2(0-1) + 1 = 2(-1) + 1 = -2 + 1 = -1\\
I_u &= 2(F_u-E_u) + 1 = 2(0 - -1) + 1 = 2(1) + 1 = 2 + 1 = 3\\
J_u &= 81(G_u+A_u) + A_u = 81(-1+1) + 1 = 81(0) + 1 = 1\\
K_u &= 40(H_u+C_u) + D_u + 1 = 40(-1+1) + 0 + 1 = 40(0)+1 = 0 + 1 = 1\\
L_u &= 40(I_u+E_u-2) + F_u = 40(3 + -1 - 2) + 0 = 40(0) + 0 = 0\\
M_{vw} &= A_vA_w + G_w = 1\cdot 2 + -1 = 2-1 = 1
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:2:3}{}_4op1_1 = {}^{1:2:3}op1_1 &= J_uD_v + L_uC_v + 41\\
&= 1\cdot 2 + 0\cdot 1 + 41\\
&= 2 + 0 + 41\\
&= 43
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:2:3}{}_4c_{od}op1 = {}^{1:2:3}c_{od}op1 &= 2G_uD_v + (I_u-1)C_v\\
&= 2\cdot -1\cdot 2 + (3-1)\cdot 1\\
&= -4 + 2\cdot 1\\
&= -2
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:2:3}{}_4c_{ev}op1 = {}^{1:2:3}c_{ev}op1 &= 2A_uD_v + (E_u-1)C_v + 1\\
&= 2\cdot 1\cdot 2 + (-1 - 1)\cdot 1 + 1\\
&= 4 + -2 + 1\\
&= 3
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:2:3}{}_{4}op2_{1} &= [2L_uA_v + J_u(2B_v+1)]D_w + [L_uE_v + J_u(F_v-1) + 41]C_w + K_u \\
&= [2\cdot 0\cdot 1 + 1(2\cdot 1 + 1)]2 + [0\cdot -1 + 1(0-1) + 41]4 + 1 \\
&= [0 + 1(3)]2 + [0 + 1(-1) + 41]4 + 1 \\
&= [3]2 + [0 - 1 + 41]4 + 1 \\
&= 6 + 40\cdot 4 + 1 \\
&= 6 + 160 + 1 \\
&= 167
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:2:3}{}_{4}z &= (G_u + A_u)M_{vw} + G_vA_w \\
&= (-1 + 1)1 + 1\cdot 2 \\
&= 0\cdot 1 + 2 \\
&= 2
\end{aligned}
$$

Checking the Calculations Against the Chart

Now, let us check these results against the chart for family ${}^{1:2:3}{}_{4}S$, Chart 21. "Well," you say, "the chart only contains the data for the first three series in the family while our example involves the fourth series." No problem. We need just extend the data from the first three columns across to a fourth column, as is shown in Chart 69.

The calculations match the chart.

Five ${}^{1:9:6}{}_{3}S$ Parameters

Our second example's results can be checked against an existing chart.

${}^{1:9:6}{}_{3}S$:

$$
\begin{aligned}
u &= 9 \\
v &= 6 \\
w &= 3 \\
A_u &= \lfloor u\div 2\rfloor = \lfloor 9\div 2\rfloor = \lfloor 4.5\rfloor = 4 \\
A_v &= \lfloor v\div 2\rfloor = \lfloor 6\div 2\rfloor = \lfloor 3\rfloor = 3 \\
A_w &= \lfloor w\div 2\rfloor = \lfloor 3\div 2\rfloor = \lfloor 1.5\rfloor = 1 \\
B_u &= \lceil u\div 2\rceil - 1 = \lceil 9\div 2\rceil - 1 = \lceil 4.5\rceil - 1 = 5 - 1 = 4 \\
B_v &= \lceil v\div 2\rceil - 1 = \lceil 6\div 2\rceil - 1 = \lceil 3\rceil - 1 = 3 - 1 = 2 \\
B_w &= \lceil w\div 2\rceil - 1 = \lceil 3\div 2\rceil - 1 = \lceil 1.5\rceil - 1 = 2 - 1 = 1 \\
C_u &= A_u^2 = 4^2 = 16 \\
C_v &= A_v^2 = 3^2 = 9 \\
C_w &= A_w^2 = 1^2 = 1 \\
D_u &= B_u(B_u+1) = 4(4+1) = 4(5) = 20 \\
D_v &= B_v(B_v+1) = 2(2+1) = 2(3) = 6 \\
D_w &= B_w(B_w+1) = 1(1+1) = 1(2) = 2 \\
E_u &= A_u(A_u-2) = 4(4-2) = 4(2) = 8 \\
E_v &= A_v(A_v-2) = 3(3-2) = 3(1) = 3
\end{aligned}
$$

$$F_u = B_u(B_u-1) = 4(4-1) = 4(3) = 12$$
$$F_v = B_v(B_v-1) = 2(2-1) = 2(1) = 2$$
$$G_u = 2(B_u-A_u) + 1 = 2(4-4) + 1 = 2(0) + 1 = 0 + 1 = 1$$
$$G_v = 2(B_v-A_v) + 1 = 2(2-3) + 1 = 2(-1) + 1 = -2 + 1 = -1$$
$$G_w = 2(B_w-A_w) + 1 = 2(1-1) + 1 = 2(0) + 1 = 0 + 1 = 1$$
$$H_u = 2(D_u-C_u) + 1 = 2(20-16) + 1 = 2(4) + 1 = 8 + 1 = 9$$
$$I_u = 2(F_u-E_u) + 1 = 2(12-8) + 1 = 2(4) + 1 = 8 + 1 = 9$$
$$J_u = 81(G_u+A_u) + A_u = 81(1+4) + 4 = 81(5) + 4 = 405 + 4 = 409$$
$$K_u = 40(H_u+C_u)+D_u+1 = 40(9+16)+20+1 = 40(25)+21 = 1000+21 = 1021$$
$$L_u = 40(I_u+E_u-2)+F_u = 40(9+8-2) + 12 = 40(15)+12 = 600+12 = 612$$
$$M_{vw} = A_vA_w + G_w = 3\cdot1 + 1 = 3+1 = 4$$

$$\begin{aligned} {}^{1:9:6}{}_{3}op1_1 = {}^{1:9:6}op1_1 &= J_uD_v + L_uC_v + 41 \\ &= 409\cdot6 + 612\cdot9 + 41 \\ &= 2454 + 5508 + 41 \\ &= 8003 \end{aligned}$$

$$\begin{aligned} {}^{1:9:6}{}_{3}C_{od}op1 = {}^{1:9:6}C_{od}op1 &= 2G_uD_v + (I_u-1)C_v \\ &= 2\cdot1\cdot6 + (9-1)9 \\ &= 12 + 8(9) \\ &= 12 + 72 \\ &= 84 \end{aligned}$$

$$\begin{aligned} {}^{1:9:6}{}_{3}C_{ev}op1 = {}^{1:9:6}C_{ev}op1 &= 2A_uD_v + (E_u-1)C_v + 1 \\ &= 2\cdot4\cdot6 + (8-1)9 + 1 \\ &= 48 + 7(9) + 1 \\ &= 48 + 63 + 1 \\ &= 112 \end{aligned}$$

$$\begin{aligned} {}^{1:9:6}{}_{3}op2_1 &= [2L_uA_v + J_u(2B_v+1)]D_w + [L_uE_v + J_u(F_v-1) + 41]C_w + K_u \\ &= [2\cdot612\cdot3+409(2\cdot2+1)]2+[612\cdot3+409(2-1)+41]1+1021 \\ &= [3672+409\cdot5]2+[1836+409+41]1+1021 \\ &= (3672+2045)2 + 2286\cdot1 + 1021 \\ &= (5717)2 + 2286 + 1021 \\ &= 11434 + 3307 \\ &= 14741 \end{aligned}$$

$$\begin{aligned} {}^{1:9:6}{}_{3}z &= (G_u + A_u)M_{vw} + G_vA_w \\ &= (1+4)4 + -1\cdot1 \\ &= 5\cdot4 - 1 \\ &= 20 - 1 \\ &= 19 \end{aligned}$$

Checking the Results

Again, let us check our results against the chart, this time Chart 66. Lo and behold, the calculated results match the chart figures.

op2 Spacing

Again, all that is missing in order for us to calculate any factor pair in this family of planes are formulas for the op2 spacing coefficients.

±

An avenue into cracking the puzzle of op2 spacing that seemed promising and that was pursued successfully was that of ± analysis. Yet another startling pattern was discovered, this time concerning the op2 spacing coefficients.

Yet Another Startling Pattern -- The Common Divisor of op2 Spacing Coefficients

Take a look at any of the family charts. Notice that besides adding up to a perfect square, the two op2 spacing coefficients for a series always have a common denominator. It just so happens that that common divisor in each case is z. So, there is an intimate relationship between z, the two individual coefficients, which are both multiples of z, and their sum, which is z^2. Furthermore, the two numbers remaining when the two coefficients are divided by z are equidistant from ½z.

An Example

As an example, let us take the op2 spacing coefficients for the series that we just worked out other parameters for, ${}^{1:9:6}{}_{3}S$. Its z, as we calculated, is 19. Its odd and even op2 spacing coefficients are 152 and 209, respectively. 152 + 209 = 361, which is 19^2. That much we have known for quite a while. What we are illustrating here, however, is that both 152 and 209 are divisible by 19. 152 = 8•19 and 209 = 11•19. Also, 8 and 11 are equidistant from ½(19) or 9½. In other words, 8 and 11 are 9½ ± 1½ (technically, 9½ ± -1½ respectively).

½z ± the Amount Called "±"

That ± amount applied to ½z is what we shall call the ± parameter for a series. Its significance is that since we can calculate z for a series, and therefore ½z, if we only knew the ± for the series, then we could calculate both op2 spacing coefficients. The odd coefficient is $z(\frac{1}{2}z + \pm)$ and the even coefficient is $z(\frac{1}{2}z - \pm)$.

Finding ± vs. op2 Spacing

Is this any better than just figuring out directly what the two coefficients are, the way the other formulas were found? It may be slightly more indirect, but the numbers to work with in finding the generalizable pattern, the ± values, are much smaller than the op2 spacing coefficients that they lead to. This is very helpful when trying to derive formulas to cover masses of data. The situation is remotely analogous to that of working with logarithms rather than with the numbers that the logarithms represent.

The Formula for ±

So, the road to formulation traveled here was the one of ± analysis. The formula is:

$$^{1:u:v}{}_{w}\pm = N_u M_{vw} - \tfrac{1}{2} G_v A_w.$$

Illustrating Its Use

Let us illustrate its use:

$$^{1:9:6}{}_{3}\pm = N_u M_{vw} - \tfrac{1}{2} G_v A_w$$

We already have values for all of these terms except for N_u, so all that we need to do is to calculate N_u:

$$\begin{aligned} N_u &= \tfrac{1}{2}(3G_u - A_u) \\ &= \tfrac{1}{2}(3 \cdot 1 - 4) \\ &= \tfrac{1}{2}(3 - 4) \\ &= \tfrac{1}{2}(-1) \\ &= -\tfrac{1}{2} \end{aligned}$$

Therefore $^{1:9:6}{}_{3}\pm = -\tfrac{1}{2} \cdot 4 - \tfrac{1}{2} \cdot -1 \cdot 1$
$= -2 - -\tfrac{1}{2}$
$= -1\tfrac{1}{2}$.

Finding Any Factor Pair in Any First-plane Child Plane

Given the formula for ±, op2 spacing odd and even coefficients can be calculated, and therefore any factor pair in any series in any family in any plane that is a child of the first plane.

What About Grandchild Planes and Beyond?

But, what about grandchild planes? Do they exist? Are there in fact an infinite number of levels, or generations, of planes?

First, an Example of Finding a Factor Pair

We shall be getting to those questions shortly. First, let us work through an example of calculating a specific factor pair now that we have a handle on op2 spacing, at least for first-plane child planes.

${}^{1:9:6}{}_{3}S_{12}$

Continuing to work with ${}^{1:9:6}{}_{3}S$, let us find the twelfth factor pair in the series, i.e. ${}^{1:9:6}{}_{3}S_{12}$. We need to find ${}^{1:u:v}{}_{w}op1_{x}$ and ${}^{1:u:v}{}_{w}op2_{x}$ for x = 12. They are derived, respectively, by adding appropriate amounts to ${}^{1:u:v}{}_{w}op1_{1}$ and ${}^{1:u:v}{}_{w}op2_{1}$. These amounts involve the odd and even spacing coefficients for the two sequences (the op1 and op2 sequences).

The Two Operands (Factors)

The expressions for the two factors in the factor pair are:

$${}^{1:u:v}{}_{w}op1_{x} = {}^{1:u:v}op1_{1} + {}^{1:u:v}c_{od}op1 \cdot C_{x} + {}^{1:u:v}c_{ev}op1 \cdot D_{x}$$

$${}^{1:u:v}{}_{w}op2_{x} = {}^{1:u:v}{}_{w}op2_{1} + {}^{1:u:v}{}_{w}c_{od}op2 \cdot C_{x} + {}^{1:u:v}{}_{w}c_{ev}op2 \cdot D_{x}$$

op1

Taking advantage of the calculations that we have already performed for ${}^{1:9:6}{}_{3}S$, we can proceed quickly to:

$$\begin{aligned}{}^{1:9:6}{}_{3}op1_{12} = {}^{1:9:6}op1_{12} &= {}^{1:9:6}op1_{1} + {}^{1:9:6}c_{od}op1 \cdot C_{12} + {}^{1:9:6}c_{ev}op1 \cdot D_{12} \\ &= 8003 + 84 \cdot C_{12} + 112 \cdot D_{12}\end{aligned}$$

$$
\begin{aligned}
C_{12} &= A_{12}^{2} \\
&= \lfloor 12 \div 2 \rfloor^{2} \\
&= \lfloor 6 \rfloor^{2} \\
&= 6^{2} \\
&= 36
\end{aligned}
$$

$$
\begin{aligned}
D_{12} &= B_{12}(B_{12}+1) \\
&= (\lceil 12 \div 2 \rceil - 1)\lceil 12 \div 2 \rceil \\
&= (\lceil 6 \rceil - 1)\lceil 6 \rceil \\
&= (6-1)6 \\
&= 5 \cdot 6 \\
&= 30
\end{aligned}
$$

So, $$
\begin{aligned}
{}^{1:9:6}op1_{12} &= 8003 + 84 \cdot 36 + 112 \cdot 30 \\
&= 8003 + 3024 + 3360 \\
&= 14387
\end{aligned}
$$

op2

$$
\begin{aligned}
{}^{1:9:6}{}_{3}Sop2_{12} &= {}^{1:9:6}{}_{3}op2_{1} + {}^{1:9:6}{}_{3}C_{od}op2 \cdot C_{12} + {}^{1:9:6}{}_{3}C_{ev}op2 \cdot D_{12} \\
&= 14741 + 152 \cdot 36 + 209 \cdot 30 \\
&= 14741 + 5472 + 6270 \\
&= 26483
\end{aligned}
$$

The Pair, Verified

So, our factor pair is 14,387 • 26,483. Double-checking that this does indeed yield an output of $x^2 + x + 41$, 14,387 • 26,483 = 381,010,921. This checks out, via the $\frac{1}{2}(4mn-163)^{\frac{1}{2}}-1$ formula, to be f(19,519).

Are All Operands Prime?

One question that might well be asked at this point, if it has not already come to mind at any of many prior points, is "Are the two factors in a factor pair both prime numbers?" In general, no. Neither number need be prime. In this example, the first factor is prime while the second is not.

Calculator Limits Reached

Another point brought to mind is the fact that as the numbers being worked with in this investigation increased in size into the hundreds of millions, the limits of my 8-digit-readout pocket calculator were reached. Recall that only in the very early stages of this study was a computer employed. Although I had my own personal computer that I could have used, these studies involve more analysis and synthesis than number crunching. Also, computers are notorious for approximating large numerical values rather than getting them exact. So, a pocket calculator was actually handier for my math needs than a computer, anyway.

A New Calculator

When I reached my calculator's limits I headed out to get a new one with a more capacious display. The one I settled on, a Casio FX-4000P, not only had an 11-digit display, but it was loaded with advanced functions as well, whereas my old calculator had only the simplest functions such as addition, subtraction, multiplication, and division. The Casio is programmable, as well, although I had no thought of using its advanced functions or its programmability.

Advanced Features

Well, I fell in love with my new toy. Not only was it great at handling much larger whole numbers and preserving their wholeness, but I decided to dabble with the program capability as well.

Programming

The First Program

I started with a simple program to calculate which input x to $x^2 + x + 41$ produced the output equal to the product of the two factors in a given factor pair. In other words, I implemented the $\frac{1}{2}(4ab-163)^{\frac{1}{2}}-1$ formula in a small program. See Appendix B for the program listing (program 1).

Programming Language

The Casio has its own programming language. In it, symbols count as "steps." As many as 10 programs can reside in memory at one time, but the total number of steps for all programs combined cannot exceed 550. Program 1 consumes 37 steps (the character string "Mcl" is a single symbol, standing for "memory clear," and thus counts as only one "step.").

The Second Program

The second program was more ambitious. Given an initial absolute x and the next op1 in a factor pair series, it finds that next op1's absolute x and its op2. To illustrate, given x = 19,519 (recalling that ${}^{1:9:6}{}_{3}S_{12}$ = f(19,519)) and the next op1 (${}^{1:9:6}op1_{13}$) = 15,731, the program determines that (${}^{1:9:6}op2_{13}$) = 28,991 and that the absolute x yielding 15,731 • 28,991 as an output is 21,355. This procedure is useful for stepping through a series without knowing the op2 spacing. Program 2 uses up 66 steps. See Appendix B for the program listing.

The Third Program

Once the formulas needed for calculating any factor pair in any first-plane child plane were known, the most ambitious Casio program of all was undertaken. The third program finds and displays, for any input u, v, and w, the $op1_1$, $c_{od}op1$, $c_{ev}op1$, z, $op2_1$, ±, $c_{od}op2$, and $c_{ev}op2$. At that point, the user is prompted for a relative x (that is, an x within the series, given that x = 1 for the first factor pair in the series). The program then promptly produces $op1_x$ and $op2_x$. The user can then enter another x and repeat the process any number of times for that series. The third program requires 509 steps of memory, and this with every coding shortcut imaginable. Even so, there is not enough room for all three programs to co-reside, so the second program was erased. The listing for the third program appears in Appendix B.

The Fourth Program

The fourth program is a slimmed-down variation on the second program. This program finds and displays the next op2 given the next op1 and the current op2. Of what value is such a program? Doesn't the program description imply that one already knows the next op1, and if so, where did that knowledge come from and why couldn't one just as easily know the next op2 without this program?

The program's great value arises because it is often much easier to know the op1's for a factor pair series than to know the op2's. This is due to the fact that all of the op1 sequences for a whole family of series are the same. To bring this fact home, look at any of the columns in either Table 63 or Table 70.

These series family columns each have the same $op1_1$ for each series starting pair within the family. They each have the same op1 sequence in general. If one knows the op1 sequence for any of the series in a family, then one knows the sequence for any of the other series in the family. Not only that -- since the op1 sequence is the same as a corresponding op2 sequence in the family's parent family (the parent plane family that generates the plane containing the family in question), the sequence may already be familiar from before, when the parent family may have been worked with.

On the other hand, each series's op2 sequence is different within a family, and there is a good chance that the op2 sequence has never been worked out before. So, if one has a handle on the op1 sequence for a series, then with the aid of the fourth program the op2's can be quickly calculated dynamically.

Roller Coaster Series

One oddity about factor pair series that has been taken into account in the fourth program is that for some series the factors in factor pairs do not strictly increase in magnitude with each successive pair. In some series the values go up and down alternately. This roller coaster effect occurs when either the odd x spacing or the even x spacing is negative. For example, take the series ${}^{1:2:3}{}_{4}S$. Its factor pairs are:

Relative x	Factors Pairs
1	43 • 167
2	41 • 163
3	47 • 179
4	41 • 167
5	53 • 199
6	43 • 179
7	61 • 227
8	47 • 199

etc.

Table 79. Roller coaster factor pairs of series ${}^{1:2:3}{}_{4}S$.

One of the inputs to the fourth program is a "direction" parameter. This tells the program to look for the next op2 either as a larger number than the current op2, or as a smaller number. The direction parameter takes either a "1" or a "-1" as its value, for go "up" or "down" respectively, when looking for the next op2. The value is used directly as an increment or decrement. If the next op1 value is less than the previous one, the direction should be -1. If the next op1 value is greater than the previous one, the direction should be 1.

Cutting Corners

In order to save steps in the fourth program so that it would not crowd the third program, certain corners were cut. Not only are the prompts for the input values omitted, but the first program was erased as well. The fourth program takes three inputs initially, which are manually placed in memory registers prior to execution of the program. The inputs are: the next op1, the current op2, and the direction.

After the initial run of the program, only the next op1 parameter must be entered manually to continue moving "up" through successive factor pair results, as long as the memory registers with the other two parameters are intact from the initial run, which they are if one runs no other program which uses those registers. Even powering off the calculator does not cause the registers to lose their data.

If the series is a roller coaster series, however, then one must enter the direction parameter as well for each calculation, as the direction keeps alternating for such a series.

Occasionally, the op2 returned as the answer is not the next one, but the current one. This occurs when by coincidence the product of the next op1 and the current op2 *also* yields an output of $x^2 + x + 41$ (the current op1 and current op2 yield such an output by definition). When the answer returned is the current op2, the correct next op2 will be returned after pressing the EXE key again.

Investigation into Grandchild Planes

The ${}^{1:5:5:6}{}_{5}S$ First Operands

Before the third program was written, investigation had already begun into the possibility of grandchild planes. ${}^{1:5:5}S$ was chosen to be expanded into a prospective grandchild plane:

${}^{1:5:5}S$	-1346• 1713•	-1632• 2285•	-1632• 5711•	-1346• 6283•	2080• 9709•	2938• 10281•
1999• 367	367•1999	653•1999	4079• 1999	4937• 1999	11789• 1999	13219• 1999
1999• 653	367• 653	653• 367	4079• 367	4937• 653	11789• 4079	13219• 4937
1999• 4079	367•4079	653•4937	4079•11789	4937•13219	11789•23497	13219•25499
1999• 4937	367• 41	653• 41	4079• 6893	4937• 9181	11789•29737	13219•34313
1999•11789	367•6893	653•9181	4079•29737	4937•34313	11789•68573	13219•75437
1999•13219	367• 163	653•1021	4079•21577	4937•27583	11789•78973	13219•90127
1999•23497						
1999•25499						

Table 80. ${}^{1:5:5}S$ expanded into a prospective grandchild plane.

With the help of the second program, a test of prospective series ${}^{1:5:5:6}{}_{6}S$ was made. This is the series that would start with 13219•90127, the sixth (prospective) starter pair in the sixth column of Table 80. These two factors do yield a viable factor pair: 13219•90127 = 1,191,388,813 = f(34,516).

Do They Really Start a Series?

The question is, does this factor pair really begin a series of factor pairs?

op1 Spacing from Parent op2 Spacing

Because of another regularity noticed in first-plane child planes, that ${}^{1:u:v}op1_1 \equiv {}^{1:u}{}_{v}op2_1$, we could work with an op1 spacing in the new plane level (grandchild level) based on an op2 spacing from the parent child-plane column. So, the op1 spacing here for column ${}^{1:5:5:6}S$ should equal the op2 spacing for parent column ${}^{1:5:5}S$ series 6, or the ${}^{1:5:5}{}_{6}S$ op2 spacing. This equals 180•, 144•.

The Second Program in Action

Given $op1_1$ and $op2_1$, and the op1 spacing, with the help of the second Casio program we could try to find out if ${}^{1:5:5:6}{}_6S$ is really a series, and if so, what its x spacing and op2 spacing are. The results support 13,219•90,127 as a series starter:

absolute x	op1		op2	
34,516	13,219	•	90,127	via Casio
469•1	180•1		1222•1	
34,985	13,399	•	91,349	
377•2	144•2		987•2	
35,739	13,687	•	93,323	
469•3	180•3		1222•3	
37,146	14,227	•	96,989	via extrapolation
377•4	144•4		987•4	
38,654	14,803	•	100,937	

Table 81. The series started by 13,219 • 90,127.

Assault on the Third Level

With these encouraging results in, the assault on the third level of planes began in earnest. $op1_1$'s and $op2_1$'s and op1 spacings would be straightforward to calculate. The only real challenge would be to find the op2 spacing for a factor pair series. This could be done via finding the ± for the series, as was done with the second-level planes.

Data for Deriving a ± Formula

What was needed in order to come up with a formula for ± was a lot of sample data, in other words many examples of op2 spacings or ±'s for third-level series from which to generalize.

Generating Data in Bulk

The problem became one of generating such test data in bulk. Methods for mass producing ± data were developed, to be described shortly, which made the goal attainable. These methods involved the discovery of shortcuts that drastically reduced the time and effort to extract the ± patterns for whole planes in assembly-line fashion. To appreciate how important this was, recall that there is a whole plane full of series for each family column in each parent plane. So, if we had looked at nine planes at level two, each with six families, then each of those 54 families could spawn a whole plane at level three. And, we might be interested in six families of three series each for each of those 54 grandchild planes. And, remember, all of this work was being done by hand.

The Streamlined Technique

To illustrate the streamlined technique that evolved for mass ± determination, let us look at an example. The example case to be worked with is for the plane generated from second-level column ${}^{1:5:4}S$. Its child plane is denoted as ${}^{1:5:4:v}S$.

Plane ${}^{1:5:4:v}S$

The First Three Series-starting Pairs

First, we need to generate the plane in the familiar way, as columns of series-starters. One shortcut here is that we only generate the first three starting pairs in each column, as the rest of them for a column can be extrapolated from the first three:

${}^{1:5:4}S$	${}^{1:5:4:v}S$					
1019• 367	367•1019	163•1019	2609• 1019	1997• 1019	6889• 1019	5869• 1019
1019• 163	367• 163	163• 367	2609• 367	1997• 163	6889• 2609	5869• 1997
1019• 2609	367•2609	163•1997	2609• 6889	1997• 5869	6889•13207	5869•11779
1019• 1997						
1019• 6889						
1019• 5869						
1019•13207						
1019•11779						

Table 82. Child plane column starts generated from ${}^{1:5:4}S$.

op2 Spacing Coefficients Divided by z, and ±

We now need to create a similarly laid-out table but one with ± data for each factor pair series, in the position where in Table 82 the series' starting factor pair appears. The exact form of the ± data in each such spot is:

$$\begin{matrix} c_{od}op2 \div z \\ c_{ev}op2 \div z \end{matrix} \quad \cdot\ z \qquad \pm$$

The intermediate data $c_{od}op2 \div z$ and $c_{ev}op2 \div z$ tie into $c_{od}op2$ and $c_{ev}op2$ but save us from writing those numbers down, too. Given these two items, and writing z next to them, we can mentally divide z by 2 and subtract that from $c_{od}op2 \div z$ to get the ± value for the series. We can also go in the other direction and reconstruct $c_{od}op2$ and $c_{ev}op2$ if we wish, by having z right there.

Let us illustrate these points. For our example we can use series $^{1:9:6}{}_{3}S$. Recall that its op2 spacing values are 152 and 209 for odd spacing and even spacing respectively. Also, z for the series is 19. We also worked out the ± value for the series as -1.5. The ± entry for the series would appear as:

$$^{8}{}_{11} \cdot 19 \quad -1.5$$

portraying that the odd op2 spacing is 8 times z, or 8 • 19 = 152, that the even op2 spacing is 11 times z, or 11 • 19 = 209, that z is 19, and that the ± value is -1.5. z ÷ 2, or 19 ÷ 2 = 9.5, is a distance of 1.5 from both 8 and 11. The ± value has a minus sign because by convention, to get the ± value, we subtract z ÷ 2 from the top number, the odd op2 spacing divided by z. In this case we subtract 9.5 from 8 to get -1.5.

To reconstruct the odd and even op2 spacings given this kind of ± entry, we simply multiply z times the two values to its left. In this example that would be 19 times 8 = 152 and 19 times 11 = 209.

Just Looking Up the op2 Spacings

In order to write down $c_{od}op2 \div z$, $c_{ev}op2 \div z$, and z for each series, we need to find that information, of course. Fortunately we can now take advantage of another shortcut. Rather than have to generate the first few factor pairs for each series in order to extract the op2 spacing from that data, we can simply look up the op2 spacings back in the parent column's chart (for column $^{1:5:4}S$ that is Chart 40). This holds true for op2's in the three series written down in each column in $^{1:5:4:v}S$, since each $op2_1$ given is directly taken from the parent column in the child plane construction process. $op2_1$'s beyond the first three in a generated column, however, can be and generally are numbers *not* seen before. So, this development is also fortuitous.

As a reminder, recall that in a newly generated column, the first $op2_1$ is familiar, being the parent column's op1; the second and third $op2_1$'s are also familiar, coming directly from $op2_1$'s in the parent column. The spacings that each op2 series had in the parent plane, when the $op2_1$'s appeared in series, can be borrowed as the spacings in the generated plane. The $op1_1$ for a generated column, of course, can take its spacing in the same way, from the spacing of the parent $op2_1$ that it was taken from.

The ± Data

So, we can now populate the table of ± data, with much-reduced effort:

w	1	2	3	4	5	6
1	${}^{4}{}_{1}\bullet 5$ 1.5	${}^{4}{}_{1}\bullet 5$ 1.5	${}^{4}{}_{1}\bullet 5$ 1.5	${}^{4}{}_{1}\bullet 5$ 1.5	${}^{4}{}_{1}\bullet 5$ 1.5	${}^{4}{}_{1}\bullet 5$ 1.5
2	${}^{-2}{}_{0}\bullet -2$ -1.0	${}^{-2}{}_{-1}\bullet -3$ -0.5	${}^{2}{}_{1}\bullet 3$ 0.5	${}^{2}{}_{0}\bullet 2$ 1.0	${}^{6}{}_{2}\bullet 8$ 2.0	${}^{6}{}_{1}\bullet 7$ 2.5
3	${}^{6}{}_{2}\bullet 8$ 2.0	${}^{6}{}_{1}\bullet 7$ 2.5	${}^{10}{}_{3}\bullet 13$ 3.5	${}^{10}{}_{2}\bullet 12$ 4.0	${}^{14}{}_{4}\bullet 18$ 5.0	${}^{14}{}_{3}\bullet 17$ 5.5

Table 83. ± values for ${}^{1:5:4:}v_S$.

The ± Pattern for a Plane Given Just the ±'s for the Plane's Upper Left

We are not solely interested in the individual ± values for generated planes such as this one, however. Instead we need to find the general pattern of ± growth across each whole plane. Given that and given the starting ± values in the upper left corner of a plane, we could calculate the ± value at any coordinates within the plane.

Admittedly this technique might seem only one possibility, with little backing to sanction it over some other choice of ± patterning to deal with. This is the actual pattern found, however, in the earlier work on generalizing ± patterns into a formula for level-2 planes. That work was not discussed in its details, so for the reader this is all new, of course. The important point is that the ± value at a given point in a plane can indeed be calculated by knowing the horizontal and vertical spacing patterns for ±'s across a plane, and given the ± values for the points in the "starting" upper left corner of the plane (to add the displacements to).

An Even Shorter Way

Upper Left Corner ± Values

It turns out, however, that we do not even have to write down this much ± data for a plane in order to derive its horizontal and vertical spacing parameters. Let us first take another look at the ± data in Table 83, this time just showing the ±'s themselves:

w	v→ 1	2	3	4	5	6
1	1.5	1.5	1.5	1.5	1.5	1.5
2	-1.0	-0.5	0.5	1.0	2.0	2.5
3	2.0	2.5	3.5	4.0	5.0	5.5

Table 84. ±'s for plane $^{1:5:4:v}S$.

± Spacings

Now let us insert the spacings between values in a column into the data in Table 84:

w	v→ 1		2		3		4		5		6	
1	1.5		1.5		1.5		1.5		1.5		1.5	
		-2.5		-2.0		-1.0		-0.5		0.5		1.0
2	-1.0		-0.5		0.5		1.0		2.0		2.5	
		3.0		3.0		3.0		3.0		3.0		3.0
3	2.0		2.5		3.5		4.0		5.0		5.5	

Table 85. ±'s for plane $^{1:5:4:v}S$ with spacings.

Orderly Alternation

Now, it just so happens that, through yet another gem of a pattern, the ±'s in a family always progress with the same alternating spacings. For example, family 1 in Table 85 would progress as:

w	v=1	
1	1.5	
		-2.5
2	-1.0	
		3.0
3	2.0	
		-2.5
4	-0.5	
		3.0
5	2.5	
	etc.	

Table 86. Family ±'s progress with simple odd and even spacings.

Reverse-deriving op2 Spacing

The actual op2 spacings that Table 86 values represent can be reverse-derived from this sequence and the regular spacing of the z's:

±	z	z÷2	c_{od}op2÷z ≡ z÷2 + ±	c_{ev}op2÷z ≡ z÷2 − ±	c_{od}op2÷z & c_{ev}op2÷z	c_{od}op2 & c_{ev}op2
1.5	5	2.5	4.0	1.0	$^{4}{}_{1}$	$^{20}{}_{5}$
-1.0	-2	-1.0	-2.0	0.0	$^{-2}{}_{0}$	$^{4}{}_{0}$
2.0	8	4.0	6.0	2.0	$^{6}{}_{2}$	$^{48}{}_{16}$
-0.5	1	0.5	0.0	1.0	$^{0}{}_{1}$	$^{0}{}_{1}$
2.5	11	5.5	8.0	3.0	$^{8}{}_{3}$	$^{88}{}_{33}$

Table 87. Reverse-deriving op2 spacing from ± and z.

More ± Spacing Examples

To further illustrate this regularity in spacings between ± values in a series, here are a couple of additional examples:

w	op2 Spacing	z	op2 spacing ÷ z	z÷2	odd op2 spacing ÷ z - z÷2 (=±)	Diff.
1	$^{4}_{0}$	2	$^{2}_{0}$	1.0	1.0	
						-1.5
2	$^{0}_{1}$	1	$^{0}_{1}$	0.5	-0.5	
						2.0
3	$^{20}_{5}$	5	$^{4}_{1}$	2.5	1.5	
						-1.5
4	$^{8}_{8}$	4	$^{2}_{2}$	2.0	0.0	
						2.0
5	$^{48}_{16}$	8	$^{6}_{2}$	4.0	2.0	
						-1.5
6	$^{28}_{21}$	7	$^{4}_{3}$	3.5	0.5	

Table 88. Regularity in ± spacing for $^{1:3:3}S$.

w	op2 Spacing	z	op2 spacing ÷ z	z÷2	odd op2 spacing ÷ z - z÷2 (=±)	Diff.
1	$^{10}_{15}$	5	$^{2}_{3}$	2.5	-0.5	
						0.0
2	$^{36}_{45}$	9	$^{4}_{5}$	4.5	-0.5	
						-1.0
3	$^{152}_{209}$	19	$^{8}_{11}$	9.5	-1.5	
						0.0
4	$^{230}_{299}$	23	$^{10}_{13}$	11.5	-1.5	
						-1.0
5	$^{462}_{627}$	33	$^{14}_{19}$	16.5	-2.5	
						0.0
6	$^{592}_{777}$	37	$^{16}_{21}$	18.5	-2.5	

Table 89. Regularity in ± spacing for $^{1:9:6}S$.

Remembering the Goal

The goal here is to be able to predict the ± for a given series within a given family column within a given generated plane. If we can predict the regular ± spacings for a column, then all that we need to know is the starting ± value for that column.

First-series ± Values

It just so happens that the ± values of the first series in every column in a plane are the same. This is because the op2 spacing for those first series are all the same. This, in turn, is due to the fact that the first op2 in every column's first series is the same number, the parent column's $op1_1$. For example, in plane $^{1:5:4:v}S$, every column's first series's first op2 is 1019, the parent column's ($^{1:5:4}S$'s) $op1_1$.

How the ± Spacings Change Across a Plane

So, all that we need for predicting a particular ± in a given generated column are the ± spacings for that column. Therefore, what we need is a handle on how those spacings change from left to right and from top to bottom across the plane as we move from the upper left corner of the plane. This handle is readily available.

Even and Odd ± Spacings

Referring back to Tables 84 and 85, notice that the even ± spacing in *every* column is the same, 3.0, though the odd ± spacings change from column to column. And, keeping in mind the regularity illustrated by Table 85, *every* even ± spacing for the plane is 3.0, while every odd ± pattern across columns is the same, -2.5, -2.0, -1.0, etc., whether the spacings are between w's 1 and 2 for columns or between w's 3 and 4, or 5 and 6, etc.

What We Need

So, given this kind of regularity, which is the same for all planes, all that we need is to find the even ± spacing (which remains constant) and the odd ± spacing pattern across columns.

The Odd Spacing's Own Spacing

As might almost be expected, the odd ± spacing's internal pattern follows an odd and even rhythm. For example, for plane $^{1:5:4:v}S$, the odd ± spacings, when reading across columns, are:

-2.5
-2.0
-1.0
-0.5
0.5
1.0

Table 90. The odd ± spacings across columns for plane $^{1:5:4:v}S$.

To see *their* spacings

-2.5	
	0.5
-2.0	
	1.0
-1.0	
	0.5
-0.5	
	1.0
0.5	
	0.5
1.0	

Table 91. ${}^{1:5:4:v}S$'s odd ± spacings' own internal spacings.

Notation for Nested Spacing

To keep our bearings, now that we are talking about spacings within spacings, we can use the nested coefficient notation. The odd ± spacings in Table 91 are also known as ${}^{1:5:4:v}c_{od}\pm$ (as opposed to, for example, ${}^{1:5:4:1}c_{od}\pm$, which is -2.5, or ${}^{1:5:4:2}c_{od}\pm$, which is -2.0). The spacings for the sequence ${}^{1:5:4:v}c_{od}\pm$ are ${}^{1:5:4:v}c_{odod}\pm$ and ${}^{1:5:4:v}c_{odev}\pm$, which are 0.5 and 1.0, respectively.

We Have What We Need -- Using It

Now we have all of the information that we need for predicting any ± in plane ${}^{1:5:4:v}S$. Let us pick a target ± to predict. Suppose that we want to figure out the ± for the fourth series in the tenth column. We know the first ± in the column, since it is the same for all columns, 1.5. We need to calculate the odd ± spacing for the column, and we already know the even ± spacing, since it is always the same for the plane, 3.0. So, what is the odd ± spacing for column 10?

$${}^{1:5:4:10}{}_{4}\pm$$

By using arithmetic familiar by now,

$$\begin{aligned} {}^{1:5:4:v}c_{od}\pm &= {}^{1:5:4:1}c_{od}\pm + ({}^{1:5:4:v}c_{odod}\pm)A_v + ({}^{1:5:4:v}c_{odev}\pm)B_v \\ &= -2.5 + 0.5 \cdot A_v + 1.0 \cdot B_v. \end{aligned}$$

Since v = 10 (the tenth column),

$$A_v = A_{10} = \lfloor 10 \div 2 \rfloor = \lfloor 5 \rfloor = 5$$

and

$$B_v = B_{10} = \lceil 10 \div 2 \rceil - 1 = \lceil 5 \rceil - 1 = 5 - 1 = 4.$$

${}^{1:5:4:v}C_{od}\pm$

Therefore,

$$^{1:5:4:v}C_{od}\pm = -2.5 + 0.5 \cdot 5 + 1.0 \cdot 4 = -2.5 + 2.5 + 4.0 = 4.0.$$

${}^{1:5:4:10}{}_{w}\pm$

So, the formula for a given ± within column 10 can now be built:

$$\begin{aligned} {}^{1:5:4:10}{}_{w}\pm &= {}^{1:5:4:10}{}_{1}\pm + {}^{1:5:4:10}C_{od}\pm \cdot A_w + {}^{1:5:4:10}C_{ev}\pm \cdot B_w \\ &= 1.5 + 4.0A_w + 3.0B_w \\ &= 1.5 + 4.0A_4 + 3.0B_4 \qquad \text{for } w = 4 \\ &= 1.5 + 4.0\lfloor 4 \div 2 \rfloor + 3.0(\lceil 4 \div 2 \rceil - 1) \\ &= 1.5 + 4.0\lfloor 2 \rfloor + 3.0(\lceil 2 \rceil - 1) \\ &= 1.5 + 4.0 \cdot 2 + 3.0(2-1) \\ &= 1.5 + 8 + 3 \cdot 1 \\ &= 9.5 + 3 \\ &= 12.5 \end{aligned}$$

Now We Explain the Shorter Way

What we have so far, then, is a method for predicting the ± within one plane, the ${}^{1:5:4:v}S$ plane. What we need is a method covering *all* third-level planes. That is where the mass production of ± data for many planes comes in, and the shortcuts for getting that data. Now that the patterning of ±'s in a plane is understood, we can better explain the streamlined technique for extracting ± parameters for a plane quickly.

The Great Shortcut

For any given plane, we need to determine its starting ±, that repeats as the ± for the first series in every column, and we need the odd and even ± spacings for columns, ${}^{1:t:u:v}c_{od}\pm$ and ${}^{1:t:u:v}c_{ev}\pm$. Since the odd ± spacing is not constant, we need its own internal odd and even spacings, ${}^{1:t:u:v}c_{odod}\pm$ and ${}^{1:t:u:v}c_{odev}\pm$. The great shortcut lies in the fact that we can extract all of these parameters from just a handful of strategically selected numbers in the upper left corner of a plane's ± data.

At Most Just Seven Entries

At first glance we would only need seven ± entries, the first three columns supplying all seven entries: column 1's first three entries, column 2's first two entries, and column 3's first two entries.

${}^{1:5:4:v}$S's ±'s Again

Let us again work with plane ${}^{1:5:4:v}$S's ± entries to illustrate this:

w	v→ 1	2	3
1	1.5	1.5	1.5
2	-1.0	-0.5	0.5
3	2.0		

Table 92. The upper left corner of the ± map for ${}^{1:5:4:v}$S.

± Spacings in a Column

Now, indicating the spacings between the ± values in a column we get:

w	v→ 1		2		3	
1	1.5		1.5		1.5	
		-2.5		-2.0		-1.0
2	-1.0		-0.5		0.5	
		3.0				
3	2.0					

Table 93. ± map upper left corner with spacings shown.

The value 3.0, the spacing between column 1's second and third ± values, is the even ± spacing constant for the whole plane, ${}^{1:5:4:v}c_{ev}\pm$. The three odd ± spacings shown, -2.5, -2.0, and -1.0, are enough data for the extraction of the odd ± spacing's own internal spacings, ${}^{1:5:4:v}c_{odod}\pm$ and ${}^{1:5:4:v}c_{odev}\pm$. The change, moving across columns, between the first odd spacing (-2.5) and the second (-2.0) yields the odd internal spacing for odd ± spacing, 0.5. Similarly, the difference between the second odd ± spacing (-2.0) and the third (-1.0) gives the even internal spacing for odd ± spacing, 1.0.

The Even and Odd Spacings

So we have the even ± spacing (3.0) and the two components of the odd ± spacing (0.5 and 1.0). We also know the initial ± value for all columns, 1.5. So we have all of the information that we need in order to predict any ± value in this plane.

An Even Easier Way

There is an even easier way to get the spacing values, though, which utilizes yet another shortcut. We need only four of those seven ± entries to extract the same information as we used the seven entries for.

Down to Four Entries

We can dispense with the first entry in each of the three columns, and need only use the second and third entries in column 1, the second in column 2, and the second in column 3, as in the following example:

w	v→	1	2	3
2		-1.0	-0.5	0.5
3		2.0		

Table 94. The key four ± map upper left corner entries.

The Even and Odd Spacings

This time we take the spacings across columns in the first row (w = 2), and continue to find the spacing vertically in column 1 (between w = 2 and w = 3) to yield:

w	v→	1		2		3
2		-1.0	0.5	-0.5	1.0	0.5
		3.0				
3		2.0				

Table 95. Key ± entry spacings.

The Same Results No Coincidence

Notice that the two horizontal spacings, 0.5 and 1.0, are exactly the odd ± spacing internal spacings found using the seven ± entries. It can be easily proven that these two methods yield the same values, thanks to the fact that each columns' first ± is the same value (in this case 1.5).

Proving No Coincidence Involved

If we call column 1's first two entries a_1 and a_2, column 2's b_1 and b_2, and column 3's c_1 and c_2, we need to show that $(b_2 - b_1) - (a_2 - a_1) = b_2 - a_2$ and that $(c_2 - c_1) - (b_2 - b_1) = c_2 - b_2$. The key is that $a_1 = b_1 = c_1$. So, $(b_2 - b_1) - (a_2 - a_1) = b_2 - b_1 - a_2 + a_1 = b_2 - a_2$. Similarly, $(c_2 - c_1) - (b_2 - b_1) = c_2 - c_1 - b_2 + b_1 = c_2 - b_2$.

Extremely Streamlined Data Generation and Pattern Generalization

Armed with this extremely streamlined means of deriving the important ± parameters for a plane, the task of bulk data collection could proceed mechanically and quickly. Having gathered the data for 54 grandchild planes (one for each of the six columns in each of the first plane's nine already documented child planes), it was just a matter of tabulating the results and then generalizing them to a ± pattern holding across all 54 planes.

The Formula for Grandchild Plane ±

The generalization became the following formula:

$$^{1:t:u:v}{}_{w}\pm = (N_t A_u - \tfrac{1}{2} G_u) M_{vw} + N_t G_v A_w.$$

An Illustration

Again, let us work through an example to see this formula in action. An obvious choice for a test case is plane ${}^{1:5:4:v}S$. Let us pick family 10, series 4 within that plane, an example series that we worked with earlier. At that time we derived its ± as being 12.5. Let us see how the formula fares in arriving at that same answer:

$t = 5$
$u = 4$
$v = 10$
$w = 4$
$A_t = \lfloor t \div 2 \rfloor = \lfloor 5 \div 2 \rfloor = \lfloor 2.5 \rfloor = 2$
$A_u = \lfloor u \div 2 \rfloor = \lfloor 4 \div 2 \rfloor = \lfloor 2 \rfloor = 2$
$A_v = \lfloor v \div 2 \rfloor = \lfloor 10 \div 2 \rfloor = \lfloor 5 \rfloor = 5$
$A_w = \lfloor w \div 2 \rfloor = \lfloor 4 \div 2 \rfloor = \lfloor 2 \rfloor = 2$
$B_t = \lceil t \div 2 \rceil - 1 = \lceil 5 \div 2 \rceil - 1 = \lceil 2.5 \rceil - 1 = 3 - 1 = 2$
$B_u = \lceil u \div 2 \rceil - 1 = \lceil 4 \div 2 \rceil - 1 = \lceil 2 \rceil - 1 = 2 - 1 = 1$
$B_v = \lceil v \div 2 \rceil - 1 = \lceil 10 \div 2 \rceil - 1 = \lceil 5 \rceil - 1 = 5 - 1 = 4$
$B_w = \lceil w \div 2 \rceil - 1 = \lceil 4 \div 2 \rceil - 1 = \lceil 2 \rceil - 1 = 2 - 1 = 1$
$G_t = 2(B_t - A_t) + 1 = 2(2-2) + 1 = 2(0) + 1 = 0 + 1 = 1$
$G_u = 2(B_u - A_u) + 1 = 2(1-2) + 1 = 2(-1) + 1 = -2 + 1 = -1$
$G_v = 2(B_v - A_v) + 1 = 2(4-5) + 1 = 2(-1) + 1 = -2 + 1 = -1$
$G_w = 2(B_w - A_w) + 1 = 2(1-2) + 1 = 2(-1) + 1 = -2 + 1 = -1$
$M_{vw} = A_v A_w + G_w = 5 \cdot 2 + -1 = 10 - 1 = 9$
$N_t = \frac{1}{2}(3G_t - A_t) = \frac{1}{2}(3 \cdot 1 - 2) = \frac{1}{2}(3-2) = \frac{1}{2} \cdot 1 = \frac{1}{2}$

$${}^{1:t:u:v}{}_{w}\pm = (N_t A_u - \tfrac{1}{2} G_u) M_{vw} + N_t G_v A_w$$

$$\begin{aligned} {}^{1:5:4:10}{}_{4}\pm &= (\tfrac{1}{2} \cdot 2 - \tfrac{1}{2} \cdot -1) \cdot 9 + \tfrac{1}{2} \cdot -1 \cdot 2 \\ &= (1 - -\tfrac{1}{2}) \cdot 9 + -1 \\ &= (1\tfrac{1}{2}) \cdot 9 - 1 \\ &= 13.5 - 1 \\ &= 12.5 \end{aligned}$$

A Quick Dip Deeper Than the Grandchild Level

A Great-grandchild Family, ${}^{1:5:5:6:6}S$

As one last reassurance that the levels of planes descend endlessly, a dip down yet another level, to a "great-grandchild" plane, was undertaken. The objective of the mission was to determine whether series were actually begun by prospective series-starting factor pairs in the sixth column in plane ${}^{1:5:5:6:v}S$.

${}^{1:5:5:6:v}S$ Plane Construction

First, plane ${}^{1:5:5:6:v}S$ was constructed:

$1:5:5:6_S$	-8282• 10281•	-11220• 16157•	-11220• 36719•	-8282• 42595•	12280• 63157•	21094• 69033•
13219• 1999	1999•13219	4937•13219	25499•13219	34313•13219	75437• 13219	90127• 13219
13219• 4937	1999• 4937	4937• 1999	25499• 1999	34313• 4937	75437• 25499	90127• 34313
13219• 25499	1999•25499	4937•34313	25499•75437	34313•90127	75437•151813	90127•172379
13219• 34313						90127•235661
13219• 75437						90127•511793
13219• 90127						
13219•151813						
13219•172379						

Table 96. Plane ${}^{1:5:5:6:v}S$.

The First Operand Pairs for Column 6, Series One Through Five

Given the first three $op2_1$'s in column 6 it was easy to find the fourth and fifth $op2_1$'s, 235,661 and 511,793 (based on the $op2_1$ spacing coefficients 21094•, and 69033• apparent between $op2_1$'s one through three).

Also, the op1 spacing for all five prospective series was known (and the same for all five), being 90127's spacing in the parent column (${}^{1:5:5:6}S$) where it is an op2: 1222•, 987•.

The Beginnings of the First Three Prospective Series

Similarly fortuitous to the boon bestowed above by having the first three $op2_1$'s, having their op2 spacings from their roles in the parent column gives us their op2 spacings here as well, which in turn gives us all that we need to lay out the beginnings of each of the first three prospective series:

Absolute x		Factors of f(x)	
34,516		90,127• 13,219	
	469•1		180•1
34,985		91,349• 13,399	
	377•2		144•2
35,739		93,323• 13,687	
55,610		90,127• 34,313	
	753•1		464•1
56,363		91,349• 34,777	
	610•2		377•2
57,583		93,323• 35,531	
124,643		90,127•172,379	
	1691•1		2340•1
126,334		91,349•174,719	
	1364•2		1885•2
129,062		93,323•178,489	

Table 97. The first three series in column $^{1:5:5:6:6}S$.

The absolute-x values shown here were calculated, given the factor pairs next to them.

x Spacing and op2 Spacing

The finishing touch on this boon was the straightforward extension of these sequences for the first three (no longer prospective) series to give us the x spacing and op2 spacing for series four and five. Here is the x spacing data and the extrapolation of it to give the values for series four and five:

w→ 1		2		3		4		5
469•	284•1	753•	469•2	1691•	284•3	1975•	469•4	2913•
377•	233•1	610•	377•2	1364•	233•3	1597•	377•4	2351•

Table 98. Extension of x spacing to series four and five.

Similarly, for the op2 spacing data:

w→ 1		2		3		4		5
180•	284•1	464•	938•2	2340•	284•3	3192•	938•4	6944•
144•	233•1	377•	754•2	1885•	233•3	2584•	754•4	5600•

Table 99. Extension of op2 spacing to series four and five.

The Beginnings of Series Four and Five

So, the first three factor pairs for series four and five were simply filled in by deduction to give the following data, fulfilling the objective of the experiment on fourth-level series:

Absolute x		Factors of f(x)	
145,737		90,127•235,661	
	1975•1		3192•1
147,531		91,349•238,268	
	1597•2		2584•2
150,906		93,323•244,021	
214,770		90,127•511,793	
	2913•1		6944•1
217,683		91,349•518,737	
	2351•2		5600•2
222,385		93,323•529,937	

Table 100. Series four and five in column $^{1:5:5:6:6}S$.

z At Any Depth

Time to Find a Universal Formula for z

The time had come to try to find a set of general formulas valid across the various levels of planes. The first parameter investigated in this light was z.

z-Maps

The First Plane

A systematic mapping of z values for whole planes was begun. The z-map for the first plane appears as follows:

Parent's z→	1	0	2	1	3	2	4	3	5
41• 41	1	1	1	1	1	1	1	1	1
41• 1	0	-1	1	0	2	1	3	2	4
41• 163	2	1	3	2	4	3	5	4	6
41• 43	1	-1	3	1	5	3	7	5	9
41• 367	3	1	5	3	7	5	9	7	11
41• 167	2	-1	5	2	8	5	11	8	14
41• 653	4	1	7	4	10	7	13	10	16
41• 373	3	-1	7	3	11	7	15	11	19
41•1021	5	1	9	5	13	9	17	13	21

Table 101. z-map for the first plane.

Left-to-right Increase

Note that the -1's in column 2 have been deduced from their surroundings, in particular the patterns of z's moving horizontally across the columns. To see these patterns clearly, when moving across look at just the odd-numbered columns or at just the even-numbered columns. There seems to be a left-to-right increase in general.

Plane $^{1:2:v}S$

The z-map for plane $^{1:2:v}S$ is interesting:

Parent's z→	1	-1	1	-1	1
1•41	0	0	0	0	0
1•41	1	-1	1	-1	1
1•43	1	-1	1	-1	1
1•43	2	-2	2	-2	2
1•47	2	-2	2	-2	2
1•47	3	-3	3	-3	3
1•53	3	-3	3	-3	3

Table 102. z-map for plane $^{1:2:v}S$.

Adjusting $^{1:2:v}S$'s Formula Given $^{1:u:v}S$'s

Here, the negative values would not be deduced in the same way. Rather, once the individual formulas for z within planes $^{1:1:v}S$, $^{1:2:v}S$, $^{1:3:v}S$, $^{1:4:v}S$, and $^{1:5:v}S$ had been derived and generalization to an overall formula for the group of planes $^{1:u:v}S$ derived, the formula for $^{1:2:v}S$ had to be adjusted to conform. The more conforming formula gives the values in Table 102.

$^{1:3:v}S$

The z-map for plane $^{1:3:v}S$ is:

Parent's z→	1	1	3	3	5	5
163• 41	2	2	2	2	2	2
163• 41	-1	-1	1	1	3	3
163• 367	3	3	5	5	7	7
163• 367	0	0	4	4	8	8
163•1019	4	4	8	8	12	12
163•1019	1	1	7	7	13	13

Table 103. z-map for plane $^{1:3:v}S$.

$^{1:4:v}S$ and $^{1...1:v}S$, and $^{1:5:v}S$

Plane $^{1:4:v}S$'s z-map is identical to plane $^{1:1:v}S$'s, and for that matter plane $^{1:1:1:v}S$'s, etc. Plane $^{1:5:v}S$'s z-map is:

Parent's z→	1	2	4	5	7
367• 41	3	3	3	3	3
367• 163	-2	-1	1	2	4
367• 653	4	5	7	8	10
367•1019	-1	1	5	7	11
367•1999	5	7	11	13	17
367•2609	0	3	9	12	18
367•4079	6	9	15	18	24
367•4933	1	5	13	17	25
367•6893	7	11	19	23	31

Table 104. z-map for plane $^{1:5:v}S$.

$^{1:6:v}S$, $^{1:7:v}S$

Plane $^{1:6:v}S$'s z-map equals plane $^{1:3:v}S$'s. Plane $^{1:7:v}S$'s z-map is:

Parent's z→	1	3	5	7	9	11
653• 41	4	4	4	4	4	4
653• 367	-3	-1	1	3	5	7
653•1021	5	7	9	11	13	15
653•1999	-2	2	6	10	14	18
653•3307	6	10	14	18	22	26
653•4937	-1	5	11	17	23	29
653•6899	7	13	19	25	31	37
653•9181	0	8	16	24	32	40

Table 105. z-map for plane $^{1:7:v}S$.

Diving Down to $^{1:7:1:v}S$

We also started diving down at this point to child planes of plane $^{1:7:v}S$. Plane $^{1:7:1:v}S$'s z-map is:

Parent's z→	4	-3	5	-2	6	-1	7	0	8	1
41• 653	1	1	1	1	1	1	1	1	1	1
41• 367	3	-4	4	-3	5	-2	6	-1	7	0
41•1021	5	-2	6	-1	7	0	8	1	9	2
41• 163	7	-7	9	-5	11	-3	13	-1	15	1
41•1471	9	-5	11	-3	13	-1	15	1	17	3
41• 41	11	-10	14	-7	17	-4	20	-1	23	2
41•2003	13	-8	16	-5	19	-2	22	1	25	4
41• 1	15	-13	19	-9	23	-5	27	-1	31	3

Table 106. z-map for plane $^{1:7:1:v}S$.

Negative and Half-negative z-columns

An interesting pattern to note here and elsewhere is the occurrence of z-map columns containing consecutive negative values having parent z-values that are negative. And, those columns with alternating negative and positive values have a parent z of zero.

$^{1:7:2:v}S$

Here is plane $^{1:7:2:v}S$'s z-map:

Parent's z→	4	-1	7	2	10	5	13	8
367• 653	3	3	3	3	3	3	3	3
367• 41	1	-4	4	-1	7	2	10	5
367•1999	7	2	10	5	13	8	16	11
367• 163	5	-5	11	1	17	7	23	13
367•4079	11	1	17	7	23	13	29	19
367•1019	9	-6	18	3	27	12	36	21
367•6893	15	0	24	9	33	18	42	27
367•2609	13	-7	25	5	37	17	49	29

Table 107. z-map for plane $^{1:7:2:v}S$.

z-spacing Maps

z-spacing Maps for These Same Planes

With this data to work with, it was now time to look for patterns across these planes and across the few levels so far represented. The next analysis was to extract the spacings between z's in a plane, to form a z-spacing map for each plane. The z-spacing maps for the planes whose z-maps comprise Tables 101-107 are:

Parent's z→	1	0	2	1	3	2	4
	-1	-2	0	-1	1	0	2
	2	2	2	2	2	2	2
	-1	-2	0	-1	1	0	2
	2	2	2	2	2	2	2

Table 108. z-spacings for $^{1:v}S$. $z(w_1) = 1$.

Parent's z→	1	-1	1	-1	1
	1	-1	1	-1	1
	0	0	0	0	0
	1	-1	1	-1	1
	0	0	0	0	0

Table 109. z-spacings for ${}^{1:2:v}S$. $z(w_1) = 0$.

Parent's z→	1	1	3	3	5
	-3	-3	-1	-1	1
	4	4	4	4	4
	-3	-3	-1	-1	1
	4	4	4	4	4

Table 110. z-spacings for ${}^{1:3:v}S$. $z(w_1) = 2$.

Parent's z→	1	2	4	5	7
	-5	-4	-2	-1	1
	6	6	6	6	6
	-5	-4	-2	-1	1
	6	6	6	6	6

Table 111. z-spacings for ${}^{1:5:v}S$. $z(w_1) = 3$.

Parent's z→	1	3	5	7	9
	-7	-5	-3	-1	1
	8	8	8	8	8
	-7	-5	-3	-1	1
	8	8	8	8	8

Table 112. z-spacings for ${}^{1:7:v}S$. $z(w_1) = 4$.

Parent's z→	4	-3	5	-2	6	-1	7	0	8
	2	-5	3	-4	4	-3	5	-2	6
	2	2	2	2	2	2	2	2	2
	2	-5	3	-4	4	-3	5	-2	6
	2	2	2	2	2	2	2	2	2

Table 113. z-spacings for ${}^{1:7:1:v}S$. $z(w_1) = 1$.

Parent's z→	4	-1	7	2	10	5	13	8
	-2	-7	1	-4	4	-1	7	2
	6	6	6	6	6	6	6	6
	-2	-7	1	-4	4	-1	7	2
	6	6	6	6	6	6	6	6

Table 114. z-spacings for $^{1:7:2:v}S$. $z(w_1) = 3$.

Please note that the z-spacing map for $^{1:1:v}S$ is the same as the one for $^{1:v}S$, that the one for $^{1:4:v}S$ is the same as the one for $^{1:1:v}S$, and that the one for $^{1:6:v}S$ is the same as the one for $^{1:3:v}S$. Therefore, they are not shown here.

Repeating Pair of Values

Looking at Tables 108-114, two key points become evident. First, the z-spacings for a column repeat in an alternating pair of values. For example, in column one of plane $^{1:v}S$'s z-spacing map the two values -1 and 2 repeat as the odd and even spacings endlessly. This regularity of z-spacings in a plane allows us to simplify our search for z-spacing patterns for planes to just the top two rows of z-spacings per plane.

First w's z and Parent Column's z

We also note down in a z-spacing map:

1) the z value for w=1 for the plane (recall that all columns' first z's are the same for a plane)

2) the z value of a column's parent series.

Generating the z's for a Column

Having the z value for w=1 for any column, coupled with the column's z-spacings, gives us all that we need in order to generate the z's for the column.

Column 1 of Plane $^{1:v}S$

For example, take column one of plane $^{1:v}S$ again. Given that its first w's z is 1, and given its z-spacings of -1 and 2, we can derive its z's:

1		
	-1	
0		
		2
2		
	-1	
1		
		2
3		
	-1	
2		
etc.		

Table 115. Deriving z's for column one of plane ${}^{1:v}S$.

The Parent z

The parent series' z value above a column's z-spacings leads the way to the breakthrough allowing determination of a general, simple pattern to z-spacings and therefore z's in all planes at all levels. This is the second point that becomes evident when looking at Tables 108-114.

Row 2 of a z-spacing Map

Row 2 of the z-spacings in each plane has a constant value, for example 4 in plane ${}^{1:3:v}S$. Adding row 1 to row 2 gives the parent z above each column. Another way to look at this is to say that the parent z's minus row 1 gives row 2. Yet another way to say this is that the parent z's minus row 2 gives row 1.

How This Helps

Why is noticing this deceptively simple relationship between the parent z's and row 2 such a breakthrough? Granted, we can already know the parent z's for child columns being generated, but how do we know the row 1 z-spacings and the constant row 2 spacings in the child z-spacing map without finding all of the child z's first?

The First w's z

Notice that row 2's constant z-spacing value is always twice the first w's z value for that plane. For example, again for plane ${}^{1:3:v}S$, row 2's constant z-spacing value is 4 (see Table 110), while the z value for all first series in the plane is 2 (z of ${}^{1:3:v}{}_{1}S = 2$ in Table 103).

Parent z and Constant ${}_1z$

All that we would need when coming into a plane in order to predict all of its z's would be the child columns' parent z's and the constant z value for the first series in each child column. Given that constant z for the first series in each column in a plane (${}^{\cdots v}{}_{1}z$ or ${}_{1}z$ for short), we simply double it to get row 2. We then have the even z-spacings for the plane. To get the odd z-spacings we need only subtract the row 2 constant from the columns' parent z's.

Grandparent z

So, do we know the constant z for each column's first series? Yes. It is the parent column's own parent series's z value; in other words the grandparent z value.

Examples with ${}^{1:7:1:v}S$ and ${}^{1:7:2:v}S$

To illustrate, let us look at planes ${}^{1:7:1:v}S$ and ${}^{1:7:2:v}S$. ${}^{1:7:1:v}S$'s child columns' constant first series' z is 1 (see Table 106). For plane ${}^{1:7:2:v}S$ it is 3 (see Table 107). Both planes' parent plane is ${}^{1:7:v}S$. Its z-spacing map contains its z-spacings and above the columns, *their* parent z's (see Table 112). ${}^{1:7:v}S$'s first column's parent z is 1. ${}^{1:7:v}S$'s second column's parent z is 3. These values of 1 and 3, z values from ${}^{1:7:v}S$'s parent plane ${}^{1:7}S$, are the grandparent z's of the constant z's of the first series in all of the columns of planes ${}^{1:7:1:v}S$ and ${}^{1:7:2:v}S$, respectively.

All That We Need Going Top-down

So, we now have all of the resources that we need to predict a plane's z's ahead of time as long as we start at the first plane and move down level by level to the plane of interest, storing important parameters as we go.

Some Background Facts to Set the Stage

Before we show the end-result of this development, we will first build up some background facts so that when the time comes to unveil the universal method for predicting z at any depth, the logic of its derivation will flow flawlessly and smoothly.

Expressing z

First, let us express a z within a plane as we can now see it:

$${}^{1\ldots v}{}_{w}z = {}^{1\ldots v}{}_{1}z + {}^{1\ldots v}c_{od}zA_{w} + {}^{1\ldots v}c_{ev}zB_{w}.$$

That is, z for a specific series w, within a specific column v, within a given plane, equals the first z in column v, plus the appropriate number (A_w) times the odd z-spacing for the column, plus the appropriate number (B_w) times the even z-spacing for the column.

Re-expressing z

Now, based on our deeper understanding of z-spacings and of z for the first series in all columns in a plane, we can begin to re-express the above equation:

$${}^{1\ldots v}{}_{1}z = \text{the plane's parent column's parent series's z}$$

$${}^{1\ldots v}c_{ev}z = 2 \cdot {}^{1\ldots v}{}_{1}z$$

$${}^{1\ldots v}c_{od}z = \text{v's parent series's z} - {}^{1\ldots v}c_{ev}z$$

This allows us to state:

$$\begin{aligned}
{}^{1\ldots t:u:v}{}_{w}z &= {}^{1\ldots t:u:v}{}_{1}z + ({}^{1\ldots t:u}{}_{v}z - 2 \cdot {}^{1\ldots t:u:v}{}_{1}z)A_{w} + 2 \cdot {}^{1\ldots t:u:v}{}_{1}zB_{w} \\
&= {}^{1\ldots t:u:v}{}_{1}z + {}^{1\ldots t:u}{}_{v}zA_{w} - 2 \cdot {}^{1\ldots t:u:v}{}_{1}zA_{w} + 2 \cdot {}^{1\ldots t:u:v}{}_{1}zB_{w} \\
&= 2 \cdot {}^{1\ldots t:u:v}{}_{1}zB_{w} - 2 \cdot {}^{1\ldots t:u:v}{}_{1}zA_{w} + {}^{1\ldots t:u:v}{}_{1}z + {}^{1\ldots t:u}{}_{v}zA_{w} \\
&= {}^{1\ldots t:u:v}{}_{1}z[2(B_{w} - A_{w}) + 1] + {}^{1\ldots t:u}{}_{v}zA_{w} \\
&= {}^{1\ldots t:u:v}{}_{1}zG_{w} + {}^{1\ldots t:u}{}_{v}zA_{w}
\end{aligned}$$

Since ${}^{1...t:u:v}{}_{1}z = {}^{1...t}{}_{u}z$, we can state:

$${}^{1...t:u:v}{}_{w}z = {}^{1...t}{}_{u}zG_{w} + {}^{1...t:u}{}_{v}zA_{w}.$$

An Example With ${}^{1:7:2:8}{}_{5}z$

Let us illustrate this fact via example. Let us find z for plane ${}^{1:7:2:v}S$, column 8, series 5, i.e. ${}^{1:7:2:8}{}_{5}z$.

$${}^{1:7:2:8}{}_{5}z = {}^{1:7}{}_{2}zG_{5} + {}^{1:7:2}{}_{8}zA_{5}$$

$$= {}^{1:7}{}_{2}z[2(B_5-A_5)+1] + {}^{1:7:2}{}_{8}zA_{5}$$

$$= {}^{1:7}{}_{2}z\{2[(\lceil 5\div 2\rceil -1)-\lfloor 5\div 2\rfloor]+1\} + {}^{1:7:2}{}_{8}z\lfloor 5\div 2\rfloor$$

$$= {}^{1:7}{}_{2}z\{2[(\lceil 2.5\rceil -1)-\lfloor 2.5\rfloor]+1\} + {}^{1:7:2}{}_{8}z\lfloor 2.5\rfloor$$

$$= {}^{1:7}{}_{2}z\{2[(3-1)-2]+1\} + {}^{1:7:2}{}_{8}z\cdot 2$$

$$= {}^{1:7}{}_{2}z[2(2-2)+1] + 2\cdot {}^{1:7:2}{}_{8}z$$

$$= {}^{1:7}{}_{2}z\cdot 1 + 2\cdot {}^{1:7:2}{}_{8}z$$

$$= {}^{1:7}{}_{2}z + 2\cdot {}^{1:7:2}{}_{8}z$$

For now, rather than deriving ${}^{1:7}{}_{2}z$ and ${}^{1:7:2}{}_{8}z$, we shall simply accept their values from the tables for ${}^{1:v}S$ and ${}^{1:7:v}S$, Tables 101 and 105 respectively. ${}^{1:7}{}_{2}z$ = the z for the second series in column 7 of the first plane, = 3 (Table 101, column 7, item 2). ${}^{1:7:2}{}_{8}z = 8$ (Table 105, column 2, item 8). So:

$${}^{1:7:2:8}{}_{5}z = {}^{1:7}{}_{2}z + 2\cdot {}^{1:7:2}{}_{8}z$$

$$= 3 + 2\cdot 8$$

$$= 3 + 16$$

$$= 19.$$

That this is the correct value can be verified by looking at item 5 in column 8 of Table 107.

Recursion

Could we not calculate ${}^{1:7}{}_{2}z$ and ${}^{1:7:2}{}_{8}z$ in the same way? Yes. And, we would eventually recurse back up to the first plane. We could also begin at the first plane and descend to the z of interest. This top-down approach is the heart of the method for finding z at any depth.

Top-down

In our example of ${}^{1:7:2:8}{}_{5}z$, we might start by finding ${}^{1}{}_{7}z$, then find ${}^{1:7}{}_{2}z$, then ${}^{1:7:2}{}_{8}z$, and finally ${}^{1:7:2:8}{}_{5}z$.

${}^{1}{}_{w}z$

Using the z-spacing concept, the formula for ${}^{1}{}_{w}z$, a z in "the first column," can be expressed as:

${}^{1}{}_{1}z + {}^{1}c_{od}zA_w + {}^{1}c_{ev}zB_w$.

Given ${}^{1}{}_{1}z = 1$, ${}^{1}c_{od}z = -1$, and ${}^{1}c_{ev}z = 2$ by observation of the first column,

$$\begin{aligned} {}^{1}{}_{w}z &= 1 + -1 \cdot A_w + 2B_w \\ &= 2B_w - A_w + 1 \\ &= 2B_w - 2A_w + 1 + A_w \\ &= 2(B_w - A_w) + 1 + A_w \\ &= G_w + A_w, \text{ or } A_w + G_w. \end{aligned}$$

${}^{1:v}{}_{w}z$

The formula for ${}^{1:v}{}_{w}z$ is:

$${}^{1:v}{}_{w}z = {}^{1:v}{}_{1}z + {}^{1:v}c_{od}zA_w + {}^{1:v}c_{ev}zB_w.$$

Given:

1) ${}^{1:v}{}_{1}z_{1} = {}_{1}z$, the z of "the first series," the grandparent series of ${}^{1:v}{}_{w}z$, = 1

2) ${}^{1:v}C_{ev}z = 2 \cdot {}_{1}z = 2$

3) ${}^{1:v}C_{od}z = {}^{1}{}_{v}z - {}^{1:v}C_{ev}z$

$= {}^{1}{}_{v}z - 2$

$= (G_v + A_v) - 2$

Therefore:

$${}^{1:v}{}_{w}z = 1 + [(G_v + A_v) - 2]A_w + 2B_w$$
$$= 1 + (G_v + A_v - 2)A_w + 2B_w$$
$$= 1 + (G_v + A_v)A_w - 2A_w + 2B_w$$
$$= 2B_w - 2A_w + 1 + (G_v + A_v)A_w$$
$$= G_w + (G_v + A_v)A_w.$$

Four Levels of Formula

We now have a formula for each level from "the first series" down to any level, such as the third level of plane, as follows:

${}^{1}{}_{w}z$	=	G_w	+	A_w	just derived
${}^{1:v}{}_{w}z$	=	G_w	+	$(G_v+A_v)A_w$	just derived
${}^{1:u:v}{}_{w}z$	=	${}^{1}{}_{u}zG_w$	+	${}^{1:u}{}_{v}zA_w$	general format
${}^{1:t:u:v}{}_{w}z$	=	${}^{1:t}{}_{u}zG_w$	+	${}^{1:t:u}{}_{v}zA_w$	general format

Re-expressing ${}^{1:v}{}_{w}z$

Can we re-express ${}^{1:v}{}_{w}z$, for example, so that it fits the general format that deeper z-formulas adhere to? Yes. Notice that A_w's coefficient in the formula for ${}^{1:v}{}_{w}z$, (G_v+A_v), is equivalent to ${}^{1}{}_{v}z$. Therefore, we can re-express ${}^{1:v}{}_{w}z$ as follows:

$$^{1:v}{}_{w}z = G_w + {}^{1}{}_{v}zA_w.$$

This form of expression fits nicely with the other z-formula formats.

Final Re-expression

To be perfectly consistent, we could even go so far as to say

$$^{1:v}{}_{w}z = {}_{1}zG_w + {}^{1}{}_{v}zA_w \qquad \text{and,}$$

$$^{1}{}_{v}z = G_w + {}_{1}zA_w.$$

Top-down z At Any Depth

Formula Cascade

We are now ready to generalize our top-down method of finding z at any depth. Given that "the first w" is level 1 (L1) and equals 1:

$$G_{L1} + A_{L1} = {}_{L1}z$$

$$G_{L2} + {}_{L1}zA_{L2} = {}^{L1}{}_{L2}z$$

$${}_{L1}zG_{L3} + {}^{L1}{}_{L2}zA_{L3} = {}^{L1:L2}{}_{L3}z$$

$${}^{L1}{}_{L2}zG_{L4} + {}^{L1:L2}{}_{L3}zA_{L4} = {}^{L1:L2:L3}{}_{L4}z$$

$${}^{L1:L2}{}_{L3}zG_{L5} + {}^{L1:L2:L3}{}_{L4}zA_{L5} = {}^{L1:L2:L3:L4}{}_{L5}z$$

etc.

Chart 70. z at any depth.

An Illustration with $^{1:7:2:8}{}_{5}z$

Let us illustrate the use of this cascade method with an example. For efficiency's sake, we will utilize the fact that for all x, G_x is 1 if x is odd, and -1 if x is even.

$^{1:7:2:8}{}_{5}z = ?$

$$
\begin{aligned}
{}_{1}z &= G_1 + A_1 \\
&= 1 + 0 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
{}^{1}{}_{7}z &= G_7 + {}_{1}zA_7 \\
&= 1 + 1 \cdot A_7 \\
&= 1 + 1 \cdot \lfloor 7 \div 2 \rfloor \\
&= 1 + 1 \cdot 3 \\
&= 1 + 3 \\
&= 4
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:7}{}_{2}z &= {}_{1}zG_2 + {}^{1}{}_{7}zA_2 \\
&= 1 \cdot G_2 + 4 \cdot A_2 \\
&= 1 \cdot -1 + 4 \cdot \lfloor 2 \div 2 \rfloor \\
&= -1 + 4 \cdot 1 \\
&= -1 + 4 \\
&= 3
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:7:2}{}_{8}z &= {}^{1}{}_{7}zG_8 + {}^{1:7}{}_{2}zA_8 \\
&= 4 \cdot G_8 + 3 \cdot A_8 \\
&= 4 \cdot -1 + 3 \cdot \lfloor 8 \div 2 \rfloor \\
&= -4 + 3 \cdot 4 \\
&= -4 + 12 \\
&= 8
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:7:2:8}{}_{5}z &= {}^{1:7}{}_{2}zG_5 + {}^{1:7:2}{}_{8}zA_5 \\
&= 3 \cdot G_5 + 8 \cdot A_5 \\
&= 3 \cdot 1 + 8 \cdot \lfloor 5 \div 2 \rfloor \\
&= 3 + 8 \cdot 2 \\
&= 3 + 16 \\
&= 19.
\end{aligned}
$$

± At Any Depth

± *Maps*

The next step was to try to find a similar general method or formula for ± at any depth. The study began with the preparation of ± maps for planes, such as the ± map for the first plane:

	v→ 1	2	3	4	5	6
41• 41	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
41• 1	2.0	-1.5	1.5	-2.0	1.0	-2.5
41•163	1.0	-2.5	0.5	-3.0	0.0	-3.5
41• 43	3.5	-3.5	2.5	-4.5	1.5	-5.5
41•367	2.5	-4.5	1.5	-5.5	0.5	-6.5
41•167	5.0	-5.5	3.5	-7.0	2.0	-8.5

Table 116. ± map for the first plane.

± *Spacing Maps*

The next step from there was to construct ± spacing maps. Remembering that we only need a minimal amount of information from the upper left corner of a ± map for a plane in order to find any ± in that plane, we can abbreviate ± maps and their corresponding ± spacing maps to just a few entries. For example, we can condense Table 116 to the following, yet be able to derive the rest of the ± map from the condensed version:

	w	v→ 1	2	3
41• 41	1	-0.5	-0.5	-0.5
41• 1	2	2.0	-1.5	1.5
41•163	3	1.0	-2.5	0.5

Table 117. Condensed version of the ± map for the first plane.

And, with spacings:

	w	v→ 1	2	3
41• 41	1	-0.5	-0.5	-0.5
		2.5	-1.0	2.0
41• 1	2	2.0	-1.5	1.5
		-1.0	-1.0	-1.0
41•163	3	1.0	-2.5	0.5

Table 118. Condensed ± map for the first plane with spacings.

The ± spacing map that can be made from this ± map is:

	v→ 1	2	3
${}_2w-{}_1w$	2.5	-1.0	2.0
${}_3w-{}_2w$	-1.0	-1.0	-1.0

Table 119. Condensed ± spacing map for the first plane.

A Reminder About Two Alternate Worlds

As a reminder, there are two "alternate" numberings for all series, stemming from a choice back in the first plane between using 2•, -1• or 0•, 1• as the first family's op1 spacing. Depending on which way one numbers the series' (and planes') op1 spacings and op2 spacings, two alternative "worlds" result. Each is equally viable and internally consistent. This alternate numbering aspect will be featured prominently in the discussion to follow.

Including Parent Alternate ±

One enhancement that we can make to the ± spacing map is to include the column's parent series's alternate ± value above the column's ± spacing values:

	v→ 1	2	3
Alt-v's ± →	1.5	-2.0	1.0
${}_2w-{}_1w$	2.5	-1.0	2.0
${}_3w-{}_2w$	-1.0	-1.0	-1.0

Table 120. ± spacing map with parent series's alternate ±.

Breakthrough

We are now ready to explain a breakthrough insight for ±'s analogous to the one for z's. Referring to Table 120, we can see that the odd ± spacings ($_2w-_1w$) are always the parent alternate ± minus the even ± spacings. In other words:

```
 2.5 =  1.5 - -1.0,
-1.0 = -2.0 - -1.0,
 2.0 =  1.0 - -1.0,
```

for v's 1, 2, and 3 respectively. The even ± spacings, always -1.0 in this example, are twice the ± value of the first series in each column (which is -0.5 for all columns in the first plane).

All That We Need

We therefore have all of the ingredients that we need in order to predict any ± in a plane -- the first ± in each column, the odd ± spacing, and the even ± spacing. Given that in a top-down approach such as was used with z we can know the parent alternate ±'s before we get to the plane of interest, all that we need is to be able to find the first ± for all columns in the plane of interest and we are all set. Given the parent alternate ±'s and the first ± in all columns, we can derive both the odd and the even ± spacings for the target plane.

The First ± for All Columns

Is there any way that we can tell what the first ± for all columns is? Yes. And, this is the final insight necessary so that we can construct our general cascade method for ± at any depth.

Some Data Via Formula

During the process of discovering this final relationship, some examples of ±'s of first series were collected for all columns of various sample planes, via the formula for ${}^{1:t:u:v}{}_{w}\pm$. Since we are focusing here on just the first series in columns, w equals 1 in all of the following data:

Plane	First ± in All Columns (= First Row in Plane)
$^{1:1:1:v}S$	-0.5
$^{1:1:2:v}S$	2.0
$^{1:1:3:v}S$	1.0
$^{1:2:1:v}S$	-0.5
$^{1:2:2:v}S$	-1.5
$^{1:2:3:v}S$	-2.5
$^{1:3:1:v}S$	-0.5
$^{1:3:2:v}S$	1.5
$^{1:3:3:v}S$	0.5
$^{1:4:1:v}S$	-0.5
$^{1:4:2:v}S$	-2.0
$^{1:4:3:v}S$	-3.0

Table 121. The first row of ±'s for a sample of planes.

Data Arranged into an Array

Next, we arranged these values into an array:

u	t→ 1	2	3	4
1	-0.5	-0.5	-0.5	-0.5
2	2.0	-1.5	1.5	-2.0
3	1.0	-2.5	0.5	-3.0

Table 122. First row ±'s arranged in an array.

A Striking Pattern

Next, happening to examine the "alternate" ± map for the first plane, a striking pattern revealed itself.

But First, Some Nomenclature

Alternating "Current"

Let us refer to the "world" that we have been working with to this point as the "current" orientation, whereas the other equally viable world can be called the "alternate" orientation. If we are working with current ± maps of designation $^{1:t:u:v}{}_{w}\pm$, then the current first plane ± map for them is designated as $^{1:t}{}_{u}\pm$, and the alternate ± map would be $^{1:t}{}_{u}\pm$ alt. Which numbering is current and which is alternate is strictly relative. Here is part of the map for ${}^{1:t}{}_{u}\pm$ alt:

	w	v→ 1	2	3	4
41• 41	1	1.5	1.5	1.5	1.5
41• 1	2	-2.0	0.5	-0.5	2.0
41•163	3	1.0	3.5	2.5	5.0

Table 123. Upper left corner of ± map for ${}^{1:t}{}_{u}\pm$ alt.

Alternate Grandparent Plane

The first plane, with respect to a plane of series with designation ${}^{1:t:u:v}{}_{w}S$, can be considered the grandparent plane. The "alternate" first plane just discussed is then the alternate grandparent plane of those planes whose ${}_{1}w$ ± values appear in Table 122.

± Plus Grandparent ± alt

Now that our terminology is ready, we can return to the topic of the remarkable tie-in. If we add each value Table 122 to the value in the corresponding position in Table 123 we get:

	1	2	3	4
1	1	1	1	1
2	0	-1	1	0
3	2	1	3	2

Table 124. ${}^{1:t:u:v}{}_{1}\pm$'s added to ${}^{1:t}{}_{u}\pm$ alt's.

A Handle on the First ± for All Columns

These columns of values are the beginnings of the columns of z's for the first plane (whether "current" or "alternate"). We now have a relation that we can use to find the ± for all columns' first series for a plane:

$${}^{1:t:u:v}{}_{1}\pm = {}^{1:t}{}_{u}z - {}^{1:t}{}_{u}\pm \text{ alt.}$$

z's Two Parts

With further investigation into this fascinating additional relationship between z and ±, we realize that

$${}^{1...v}{}_{w}z = {}^{1...v}{}_{w}\pm + {}^{1...v}{}_{w}\pm \text{ alt.}$$

Therefore,

$${}^{1...v}{}_{w}z - {}^{1...v}{}_{w}\pm \text{ alt} = {}^{1...v}{}_{w}\pm.$$

How to Find a ±

Let us now express how to find a ±:

$${}^{1...v}{}_{w}\pm = {}^{1...v}{}_{1}\pm + {}^{1...v}c_{od}\pm A_{w} + {}^{1...v}c_{ev}\pm B_{w}$$

$${}^{1...v}c_{od}\pm = \text{parent } \pm \text{ alt} - {}^{1...v}c_{ev}\pm$$ (recall "Breakthrough")

$$\text{parent } \pm \text{ alt} = {}^{1...u}{}_{v}\pm \text{ alt}$$

${}^{1...v}c_{ev}\pm$ = twice the ± for all columns' first series

$$= 2 \cdot {}^{1...v}{}_{1}\pm$$

So,

$${}^{1...v}c_{od}\pm = {}^{1...u}{}_{v}\pm \text{ alt} - 2 \cdot {}^{1...v}{}_{1}\pm$$

And,

$${}^{1...v}{}_{1}\pm = {}^{1...t}{}_{u}z - {}^{1...t}{}_{u}\pm \text{ alt}$$

or

$$= {}^{1...t}{}_{u}\pm$$

So,

$${}^{1\ldots v}c_{od}\pm = {}^{1\ldots u}{}_{v}\pm \text{ alt} - 2\cdot{}^{1\ldots t}{}_{u}\pm$$

And,

$${}^{1\ldots v}c_{ev}\pm = 2 \cdot {}^{1\ldots t}{}_{u}\pm$$

Finally,

$$\begin{aligned}
{}^{1\ldots v}{}_{w}\pm &= {}^{1\ldots t}{}_{u}\pm + ({}^{1\ldots u}{}_{v}\pm \text{ alt} - 2\cdot{}^{1\ldots t}{}_{u}\pm)A_w + 2\cdot{}^{1\ldots t}{}_{u}\pm B_w \\
&= {}^{1\ldots t}{}_{u}\pm + {}^{1\ldots u}{}_{v}\pm \text{ alt}A_w - 2\cdot{}^{1\ldots t}{}_{u}\pm A_w + 2\cdot{}^{1\ldots t}{}_{u}\pm B_w \\
&= 2\cdot{}^{1\ldots t}{}_{u}\pm B_w - 2\cdot{}^{1\ldots t}{}_{u}\pm A_w + {}^{1\ldots t}{}_{u}\pm + {}^{1\ldots u}{}_{v}\pm \text{ alt}A_w \\
&= {}^{1\ldots t}{}_{u}\pm[2(B_w - A_w) + 1] + {}^{1\ldots u}{}_{v}\pm \text{ alt}A_w \\
&= {}^{1\ldots t}{}_{u}\pm G_w + {}^{1\ldots u}{}_{v}\pm \text{ alt}A_w.
\end{aligned}$$

Making the Abstract Concrete

Six Tables per Level for ${}^{1:2:3:4}{}_{5}S$

Let us illustrate these abstract relationships by following the path from the first plane down to series ${}^{1:2:3:4}{}_{5}S$, showing the following tables at each step of the descent:

- series-starting factor pairs
- z's
- ±'s
- ± spacings
- alternate ±'s
- alternate ± spacings.

The First Pair -- The Apex

The very apex of this number universe is the single factor pair 41•41, "the first pair," which has no z, no ±, nor any of the other attributes that we will lay out. It is designated by ${}_{1}S_{1}$.

The First Series

The next level down, "the first series," begins with 41•41. It is designated by $_1S$:

Designation	Series-starter	z	±	± Spacing	± alt	± alt Spacing
$_1S$	41•41	1	-0.5	n/a	1.5	n/a

Table 124. The first series.

The First Family

Next comes "the first family," $^{1}_{w}S$:

Series-starters	z	±	± Spacing	± alt	± alt Spacing
41• 41	1	-0.5		1.5	
			2.5		-3.5
41• 1	0	2.0		-2.0	
			-1.0		3.0
41•163	2	1.0		1.0	
			2.5		-3.5
41• 43	1	3.5		-2.5	
			-1.0		3.0
41•367	3	2.5		0.5	
			2.5		-3.5
41•167	2	5.0		-3.0	
			-1.0		3.0
41•653	4	4.0		0.0	

Table 125. z's, ±'s, and ± alt's for the first family.

The First Plane

Next comes "the first plane," $^{1:v}S$:

41• 41	41• 41	1•41	163• 41	43• 41	367• 41	167• 41	653• 41
41• 1	41• 1	1•41	163• 41	43• 1	367• 163	167• 43	653• 367
41•163	41•163	1•43	163• 367	43•167	367• 653	167• 373	653•1021
41• 43	41• 43	1•43	163• 367	43• 47	367•1019	167• 379	653•1999
41•367	41•367	1•47	163•1019	43•379	367•1999	167•1039	653•3307
41•167	41•167	1•47	163•1019	43•179	367•2609	167•1049	653•4937
41•653	41•653	1•53	163•1997	43•677	367•4079	167•2039	653•6899
41•373	41•373	1•53	163•1997	43•397	367•4933	167•2053	653•9181

Table 126. Series-starting factor pairs in the first plane.

Parent's z→	1	0	2	1	3	2	4	3	5
41• 41	1	1	1	1	1	1	1	1	1
41• 1	0	-1	1	0	2	1	3	2	4
41• 163	2	1	3	2	4	3	5	4	6
41• 43	1	-1	3	1	5	3	7	5	9
41• 367	3	1	5	3	7	5	9	7	11
41• 167	2	-1	5	2	8	5	11	8	14
41• 653	4	1	7	4	10	7	13	10	16
41• 373	3	-1	7	3	11	7	15	11	19
41•1021	5	1	9	5	13	9	17	13	21

Table 127. z's for the first plane.

	v→ 1	2	3	4	5	6
41• 41	-0.5	-0.5	-0.5	-0.5	-0.5	-0.5
41• 1	2.0	-1.5	1.5	-2.0	1.0	-2.5
41•163	1.0	-2.5	0.5	-3.0	0.0	-3.5
41• 43	3.5	-3.5	2.5	-4.5	1.5	-5.5
41•367	2.5	-4.5	1.5	-5.5	0.5	-6.5
41•167	5.0	-5.5	3.5	-7.0	2.0	-8.5

Table 128. ±'s for the first plane.

v→	1	2	3	4	5	6
${}_{e}w - {}_{o}w$	2.5	-1.0	2.0	-1.5	1.5	-2.0
${}_{o}w - {}_{e}w$	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0

Table 129. ± spacings for the first plane.

	w	v→ 1	2	3
41• 41	1	1.5	1.5	1.5
41• 1	2	-2.0	0.5	-0.5
41•163	3	1.0	3.5	2.5

Table 130. Alternate ±'s for the first plane.

v→	1	2	3	4	5	6
${}_{e}w - {}_{o}w$	-3.5	-1.0	-2.0	0.5	-0.5	2.0
${}_{o}w - {}_{e}w$	3.0	3.0	3.0	3.0	3.0	3.0

Table 131. Alternate ± spacings for the first plane.

Child Plane $^{1:2:v}S$

Next comes child plane $^{1:2:v}S$:

1•41	41• 1	41• 1	43• 1	43• 1	47• 1	47• 1	53• 1
1•41	41• 41	41• 41	43• 41	43• 41	47• 43	47• 43	53• 47
1•43	41• 43	41• 43	43• 47	43• 47	47• 53	47• 53	53• 61
1•43	41•163	41•163	43•167	43•167	47•179	47•179	53•199
1•47	41•167	41•167	43•179	43•179	47•199	47•199	53•227
1•47	41•367	41•367	43•379	43•379	47•409	47•409	53•457

Table 132. Plane $^{1:2:v}S$.

Parent's z→	1	-1	1	-1	1
1•41	0	0	0	0	0
1•41	1	-1	1	-1	1
1•43	1	-1	1	-1	1
1•43	2	-2	2	-2	2
1•47	2	-2	2	-2	2
1•47	3	-3	3	-3	3
1•53	3	-3	3	-3	3

Table 133. z's for plane $^{1:2:v}S$.

w	v→ 1	2	3	4	5	6
1	2.0	2.0	2.0	2.0	2.0	2.0
2	-0.5	-1.5	1.5	0.5	3.5	2.5
3	3.5	2.5	5.5	4.5	7.5	6.5
4	1.0	-1.0	5.0	3.0	9.0	7.0
5	5.0	3.0	9.0	7.0	13.0	11.0
6	2.5	-0.5	8.5	5.5	14.5	11.5

Table 134. ±'s for plane $^{1:2:v}S$.

$^{1:2}{}_{v}\pm$ alt →	1.5	0.5	3.5	2.5	5.5	4.5
v→	1	2	3	4	5	6
${}_{e}W - {}_{o}W$	-2.5	-3.5	-0.5	-1.5	1.5	0.5
${}_{o}W - {}_{e}W$	4.0	4.0	4.0	4.0	4.0	4.0

Table 135. ± spacings for plane $^{1:2:v}S$.

w	v→ 1	2	3	4	5	6
1	-2.0	-2.0	-2.0	-2.0	-2.0	-2.0
2	1.5	0.5	-0.5	-1.5	-2.5	-3.5
3	-2.5	-3.5	-4.5	-5.5	-6.5	-7.5
4	1.0	-1.0	-3.0	-5.0	-7.0	-9.0
5	-3.0	-5.0	-7.0	-9.0	-11.0	-13.0
6	0.5	-2.5	-5.5	-8.5	-11.5	-14.5

Table 136. Alternate ±'s for plane $^{1:2:v}S$.

${}^{1:2}{}_{v}\pm$ →	-0.5	-1.5	-2.5	-3.5	-4.5	-5.5
v→	1	2	3	4	5	6
${}_{e}W - {}_{o}W$	3.5	2.5	1.5	0.5	-0.5	-1.5
${}_{o}W - {}_{e}W$	-4.0	-4.0	-4.0	-4.0	-4.0	-4.0

Table 137. Alternate ± spacings for plane ${}^{1:2:v}S$.

Grandchild Plane ${}^{1:2:3:v}S$

Finally, we arrive at grandchild plane ${}^{1:2:3:v}S$:

v→	1	2	3	4	5	6
43• 1	1•43	41• 43	47• 43	167• 43	179• 43	379• 43
43• 41	1•41	41• 1	47• 1	167• 41	179• 47	379• 167
43• 47	1•47	41•167	47•179	167• 379	179• 397	379• 677
43•167	1•29	41• 41	47• 53	167• 373	179• 409	379•1049
43•179	1•41	41•373	47•409	167•1049	179•1109	379•2069
43•379	1•31	41•163	47•199	167•1039	179•1129	379•2689

Table 138. Plane ${}^{1:2:3:v}S$.

W v→	1	2	3	4	5	6
1	1	1	1	1	1	1
2	-1	0	0	1	1	2
3	1	2	2	3	3	4
4	-1	1	1	3	3	5
5	1	3	3	5	5	7
6	-1	2	2	5	5	8

Table 139. z's for plane ${}^{1:2:3:v}S$.

w v→	1	2	3	4	5	6
1	-2.5	-2.5	-2.5	-2.5	-2.5	-2.5
2	0.5	2.0	-2.0	-0.5	-4.5	-3.0
3	-4.5	-3.0	-7.0	-5.5	-9.5	-8.0
4	-1.5	1.5	-6.5	-3.5	-11.5	-8.5
5	-6.5	-3.5	-11.5	-8.5	-16.5	-13.5
6	-3.5	1.0	-11.0	-6.5	-18.5	-14.0

Table 140. ±'s for plane ${}^{1:2:3:v}S$.

${}^{1:2:3}{}_{v}\pm$ alt →	-2.0	-0.5	-4.5	-3.0	-7.0	-5.5
v→	1	2	3	4	5	6
${}_{e}W - {}_{o}W$	3.0	4.5	0.5	2.0	-2.0	-0.5
${}_{o}W - {}_{e}W$	-5.0	-5.0	-5.0	-5.0	-5.0	-5.0

Table 141. ± spacings for plane ${}^{1:2:3:v}S$.

w	v→ 1	2	3	4	5	6
1	3.5	3.5	3.5	3.5	3.5	3.5
2	-1.5	-2.0	2.0	1.5	5.5	5.0
3	5.5	5.0	9.0	8.5	12.5	12.0
4	0.5	-0.5	7.5	6.5	14.5	13.5
5	7.5	6.5	14.5	13.5	21.5	20.5
6	2.5	1.0	13.0	11.5	23.5	22.0

Table 142. Alternate ±'s for plane $^{1:2:3:v}S$.

$^{1:2:3}{}_{v}\pm$ →	2.0	1.5	5.5	5.0	9.0	8.5
v→	1	2	3	4	5	6
$_eW - {}_oW$	-5.0	-5.5	-1.5	-2.0	2.0	1.5
$_oW - {}_eW$	7.0	7.0	7.0	7.0	7.0	7.0

Table 143. Alternate ± spacings for plane $^{1:2:3:v}S$.

Lineage and Descent of a ±

Armed with our tables we can now point out how the ± for a series ties in with its lineage and descent from the first plane and above.

Beginning at the Bottom: $^{1:2:3:4}{}_5S$

Let us recall that our example here is $^{1:2:3:4}{}_5S$. Working from the bottom up, refer to Table 138, the series-starting factor pairs for plane $^{1:2:3:v}S$. Our example's series-starter is therefore column 4, item 5: 167•1049. z, from Table 139, is 5. One ± is -8.5 while the other is 13.5, each being the other's alternate. Note that the sum of the two ±'s equals z, 5.

Converting ± to op2 Spacing

Just to tie this abstraction of ± back into reality, we can translate the two ±'s to op2 spacings. This involves the reversal of the process that translates op2 spacing to ±, for example:

$^{1:9:6}{}_3S$ op2 spacing	z	spacing÷z	z÷2	(odd) spacing÷z − z÷2 (= ±)
152•		8		
	19		9.5	-1.5
209•		11		

Table 144. Sample conversion of op2 spacing to ±.

In reverse, given ± and z, this would be:

(odd) ± (= spacing÷z - z÷2)	z	z÷2	± ∓ z÷2 (= spacing÷z)	$^{1:9:6}{}_{3}S$ op2 spacing
			8	152•
-1.5	19	9.5		
			11	209•

Table 145. Sample conversion of ± to op2 spacing.

A General Conversion Formula

A general formula for translating ± to op2 spacing is:

op2 spacing = $(z^2 \pm 2z \cdot \text{"}\pm\text{"}) \div 2$.

For example,

$$
\begin{aligned}
{}^{1:9:6}{}_{3}S \text{ op2 spacing} &= [19^2 \pm 2 \cdot 19 \cdot (-1.5)] \div 2 \\
&= [361 \pm 38 \cdot (-1.5)] \div 2 \\
&= (361 \pm -57) \div 2 \\
&= (304,\ 418) \div 2 \\
&= 152,\ 209 \\
&\rightarrow 152\bullet,\ 209\bullet
\end{aligned}
$$

Applying this same technique to our two ±'s for $^{1:2:3:4}{}_{5}S$:

$$
\begin{aligned}
{}^{1:2:3:4}{}_{5}S \text{ op2 spacing} &= [5^2 \pm 2 \cdot 5 \cdot (-8.5)] \div 2 \\
&= [25 \pm 10 \cdot (-8.5)] \div 2 \\
&= (25 \pm -85) \div 2 \\
&= (-60,\ 110) \div 2 \\
&= -30,\ 55 \\
&\rightarrow -30\bullet,\ 55\bullet
\end{aligned}
$$

$$
\begin{aligned}
\text{Alternate op2 spacing} &= [5^2 \pm 2 \cdot 5 \cdot (13.5)] \div 2 \\
&= [25 \pm 10 \cdot (13.5)] \div 2 \\
&= (25 \pm 135) \div 2 \\
&= (160,\ -110) \div 2 \\
&= 80,\ -55 \\
&\rightarrow 80\bullet,\ -55\bullet
\end{aligned}
$$

Notice that the sum of the odd and even op2 spacing coefficients in both cases is 25, z^2.

These op2 Spacing Values Give Two Alternative op2 Sequences

Applying these op2 spacing values in their real-world context, they give us two alternative op2 sequences:

```
1049
       -30•1
1019
                55•2
1129
       -30•3
1039
                55•4
1259
etc.
```

Table 146. One alternative op2 sequence for series ${}^{1:2:3:4}{}_{5}S$.

```
1049
       80•1
1129
              -55•2
1019
       80•3
1259
              -55•4
1039
etc.
```

Table 147. Another alternative op2 sequence for series ${}^{1:2:3:4}{}_{5}S$.

These two op2 sequences would of course be paired with appropriate op1 sequences to yield proper operand pairs (factor pairs).

± Spacings and ± alt Spacings

Returning to our main topic of cascading ± values, let us look at the ± spacing and ± alt spacing tables for ${}^{1:2:3:4}{}_{5}S$.

Even ± Spacing vs. ± for All Columns' First Series

Notice that the even ± spacing in Table 141, -5.0, is twice the ± value for all columns' first series in Table 140, -2.5. The same holds for the even ± spacing in Table 143, 7.0, vs. the ± for all columns' first series in table 142, 3.5.

Odd ± Spacing vs. Parent ± alt Minus Even ± Spacing

Furthermore, note that each odd ± spacing in Table 141 equals the parent v's ± alt value minus the even ± spacing. For example, for v=1 the odd ± spacing, 3.0, equals the parent v's ± alt, -2.0, minus the even ± spacing, -5.0, In short, 3.0 = -2.0 - (-5.0). The same rule holds for the odd ± spacings in Table 143.

All Columns' First ± vs. Grandparent ±

Finally, those ± values that are constant for all columns' first series in Tables 140 and 142 equal the grandparent series' ± values. The grandparent series for ${}^{1:2:3:4}{}_{5}S$ is ${}^{1:2}{}_{3}S$. Refer to column 2, item 3 in both Tables 128 and 130. The ± values there are -2.5 and 3.5, respectively.

Another Example

For another example of this last effect, note that the constant ± value for all columns' first series in Tables 134 and 136 are 2.0 and -2.0, respectively. These ± maps are the alternate versions for plane ${}^{1:2:v}S$. The grandparent series for series ${}^{1:2:v}{}_{w}S$ is ${}^{1}{}_{2}S$. Refer to Table 125, ${}^{1}{}_{w}S$ (the first family). Under the column for ±, item 2 is "2.0". Under ± alt, item 2 is "-2.0."

Recapping the Recursive Method for Finding Any ±

To recap, then, the general method for finding any ± value, the recursive formula

$${}^{1\ldots t:u:v}{}_{w}\pm = {}^{1\ldots t}{}_{u}\pm G_{w} + {}^{1\ldots t:u}{}_{v}\pm \text{ alt } A_{w}$$

will give the two correct alternate ± values.

A Final Note on Usage

One final note on usage follows. We define two precursor ± items:

$_{1}\pm_{1} \equiv (-0.5,\ 1.5)$
$_{1}\pm \equiv (-0.5,\ 1.5)$.

$_{1}\pm$ is at least intuitively meaningful as the ± for "the first series." $_{1}\pm_{1}$ is not as easy to interpret, since $_{1}S_{1}$ is not a series but simply a single factor pair, "the first pair," 41•41.

Re-expressing the Method as a Cascade

These two precursor items are used to express $^{1}{}_{w}\pm$ in a form consistent with the expressions for all deeper levels:

$$^{1}{}_{w}\pm = {}_{1}\pm_{1} G_{w} + {}_{1}\pm \text{ alt } A_{w}$$

$$^{1:v}{}_{w}\pm = {}_{1}\pm\ G_{w} + {}^{1}{}_{v}\pm \text{ alt } A_{w}$$

$$^{1:u:v}{}_{w}\pm = {}^{1}{}_{u}\pm\ G_{w} + {}^{1:u}{}_{v}\pm \text{ alt } A_{w}$$

and so on.

In general, again, the formula is:

$$^{1\ldots t:u:v}{}_{w}\pm = {}^{1\ldots t}{}_{u}\pm G_{w} + {}^{1\ldots t:u}{}_{v}\pm \text{ alt } A_{w}.$$

A Sample Calculation for $^{1:2:3:4}{}_{5}\pm$

Let us work through a sample calculation of a ± value. The two alternate choices of value at any given point are separated by a comma. The "alt" of any pair of values (a,b) is simply the reverse (b,a).

$$ {}_{1}\pm_{1} = {}_{1}\pm $$
$$ = (-0.5,1.5) $$

$$ {}^{1}{}_{2}\pm = {}_{1}\pm_{1}G_{2} + {}_{1}\pm \text{ alt } A_{2} $$
$$ = (-0.5,1.5)\cdot -1 + (1.5,-0.5)\cdot 1 $$
$$ = (0.5,-1.5) + (1.5,-0.5) $$
$$ = (2.0,-2.0) $$

$$ {}^{1:2}{}_{3}\pm = {}_{1}\pm G_{3} + {}^{1}{}_{2}\pm \text{ alt } A_{3} $$
$$ = (-0.5,1.5)\cdot 1 + (-2.0,2.0)\cdot 1 $$
$$ = (-0.5,1.5) + (-2.0,2.0) $$
$$ = (-2.5,3.5) $$

$$ {}^{1:2:3}{}_{4}\pm = {}^{1}{}_{2}\pm G_{4} + {}^{1:2}{}_{3}\pm \text{ alt } A_{4} $$
$$ = (2.0,-2.0)\cdot -1 + (3.5,-2.5)\cdot 2 $$
$$ = (-2.0,2.0) + (7.0,-5.0) $$
$$ = (5.0,-3.0) $$

$$ {}^{1:2:3:4}{}_{5}\pm = {}^{1:2}{}_{3}\pm G_{5} + {}^{1:2:3}{}_{4}\pm \text{ alt } A_{5} $$
$$ = (-2.5,3.5)\cdot 1 + (-3.0,5.0)\cdot 2 $$
$$ = (-2.5,3.5) + (-6.0,10.0) $$
$$ = (-8.5,13.5) $$

The First Factor Pair for Any Series

The Last Ingredients Needed -- $op1_1$ and $op2_1$

One final area of this work needs generalization before a thoroughly general method for finding factor pairs anywhere in the plane-world would be complete. We need a way to find the first factor pair for any series. Thus, we need a method for finding ${}^{1...v}{}_{w}op1_1$ and ${}^{1...v}{}_{w}op2_1$.

$op1_1$'s and $op2_1$'s for a Family

We know that the first op1 is the same for every series in a family (column), so that should be fairly straightforward to predict. What about first op2's in a family? They progress via odd and even spacing. The odd and even spacing values are available from just the first three values in an $op2_1$ sequence. Refer to Table 70, the series-starters for plane ${}^{1:5:v}S$. Note that the $op2_1$ spacings for each column appear above the column. As a memory refresher, column 1's spacings of -204• and 245• give us the $op2_1$ sequence:

$op2_1$	odd $op2_1$ spacing	even $op2_1$ spacing
367		
	-204•1	
163		
		245•2
653		
	-204•3	
41		
		245•4
1021		
etc.		

Table 148. $op2_1$ sequence for family ${}^{1:5:1}S$, with $op2_1$ spacing.

The Sum of the $op2_1$ Spacings

So, we need a method for arriving at the $op1_1$, the first $op2_1$, and the $op2_1$ spacings for a target column, in order to predict any

$op1_1/op2_1$ combination in the column (family). Another fascinating relationship that will help streamline these calculations is that the sum of the odd and even $op2_1$ spacings for a column equals the $op1_1$ value. For example, referring again to Table 70, column 1's $op1_1$ is 41; the sum of the $op2_1$ spacings is -204 + 245 = 41. The same relation holds for the rest of the columns in Table 70, and indeed for all columns in all planes.

The $op1_1$ and the First Two $op2_1$'s

So, if we know the $op1_1$ for a column and just the first two $op2_1$'s, then we have enough information to get our four goal items. We have the $op1_1$ for the column and the first $op2_1$. The odd $op2_1$ spacing is the difference between the first and second $op2_1$'s. Finally, the even $op2_1$ spacing is the difference between the $op1_1$ and the odd $op2_1$ spacing, since the odd $op2_1$ spacing plus the even $op2_1$ spacing equals the $op1_1$.

The Threading Tying Parent and Child Columns

So, how do we predict the $op1_1$ and the first two $op2_1$'s of a column? Recalling our work on generating a child plane from a parent column, let us illustrate the threading tying parent columns to child columns. This will give us the insight that we need to see how to predict child $op1_1$'s and $op2_1$'s.

The Example of Column ${}^{1:7:4}S$

Suppose that we want the $op1_1$ and the first two $op2_1$'s for column ${}^{1:7:4}S$. Let us list some information as follows:

Column ${}^{1}S$	Column ${}^{1:7}S$	Column ${}^{1:7:4}S$
41• 41	653• 41	1999• 653
41• 1	653• 367	1999• 367
41•163	653•1021	1999• 4937
41• 43	653•1999	1999• 4079
41•367	653•3307	1999•13219
41•167	653•4937	1999•11789
41•653	653•6899	1999•25499

Table 149. $op1_1$'s and $op2_1$'s for columns ${}^{1}S$, ${}^{1:7}S$, and ${}^{1:7:4}S$.

Now let us repeat this table with some arrows included to indicate key relationships from parent column to child column to grandchild column:

Column ^{1}S	Column $^{1:7}S$	Column $^{1:7:4}S$
41• 41	>653• 41	>1999• 653
41• 1	653• 367<	1999• 367<
41•163	653•1021	1999• 4937
41• 43	653•1999	1999• 4079
41•367	653•3307	1999•13219
41•167	653•4937	1999•11789
41•653	653•6899	1999•25499

Table 150. Parent, child, and grandchild connections between columns ^{1}S, $^{1:7}S$, and $^{1:7:4}S$.

Parent $op2_1$, $op1_1$, and $_{n\text{-}2}op2_1$

Notice that the $op1_1$ in child column $^{1\ldots n}S$ comes from $op2_1$ number n in the parent column. The first $op2_1$ in a child column comes from the $op1_1$ of the parent column. Finally, the second $op2_1$ in child column $^{1\ldots n}S$ comes from parent $op2_1$ number n-2.

All Set

So, we can predict the items that we need for a child column given the parent column. We are all set to specify our general method for deriving any $op1_1/op2_1$ pair:

set $\quad {}^{1}_{1}\mathrm{op2}_1 = 41$

set $\quad {}^{1}_{2}\mathrm{op2}_1 = 1$

set $\quad {}^{1}\mathrm{op1}_1 = 41$

find $\quad {}^{1}_{w-2}\mathrm{op2}_1 = {}^{1}_{1}\mathrm{op2}_1 + {}^{1}_{2}c_{od}\mathrm{op2}_1{}^{*1} \bullet C_{w-2} + {}^{1}_{2}c_{ev}\mathrm{op2}_1{}^{*2} \bullet D_{w-2}$

find $\quad {}^{1}_{w}\mathrm{op2}_1 = {}^{1}_{1}\mathrm{op2}_1 + {}^{1}_{2}c_{od}\mathrm{op2}_1 \bullet C_{w} + {}^{1}_{2}c_{ev}\mathrm{op2}_1 \bullet D_{w}$

set $\quad {}^{1:v}_{1}\mathrm{op2}_1 = {}^{1}\mathrm{op1}_1$ (previous)

set $\quad {}^{1:v}_{2}\mathrm{op2}_1 = {}^{1}_{v-2}\mathrm{op2}_1$ (three)

set $\quad {}^{1:v}\mathrm{op1}_1 = {}^{1}_{v}\mathrm{op2}_1$ (lefthand sides)

find $\quad {}^{1:v}_{w-2}\mathrm{op2}_1 = {}^{1:v}_{1}\mathrm{op2}_1 + {}^{1:v}c_{od}\mathrm{op2}_1{}^{*3} \bullet C_{w-2} + {}^{1:v}c_{ev}\mathrm{op2}_1{}^{*4} \bullet D_{w-2}$

find $\quad {}^{1:v}_{w}\mathrm{op2}_1 = {}^{1:v}_{1}\mathrm{op2}_1 + {}^{1:v}c_{od}\mathrm{op2}_1 \bullet C_{w} + {}^{1:v}c_{ev}\mathrm{op2}_1 \bullet D_{w}$

set $\quad {}^{1:u:v}_{1}\mathrm{op2}_1 = {}^{1:u}\mathrm{op1}_1$ (previous)

set $\quad {}^{1:u:v}_{2}\mathrm{op2}_1 = {}^{1:u}_{v-2}\mathrm{op2}_1$ (three)

set $\quad {}^{1:u:v}\mathrm{op1}_1 = {}^{1:u}_{v}\mathrm{op2}_1$ (lefthand sides)

find $\quad {}^{1:u:v}_{w-2}\mathrm{op2}_1 = {}^{1:u:v}_{1}\mathrm{op2}_1 + {}^{1:u:v}c_{od}\mathrm{op2}_1{}^{*5} \bullet C_{w-2} + {}^{1:u:v}c_{ev}\mathrm{op2}_1{}^{*6} \bullet D_{w-2}$

find $\quad {}^{1:u:v}_{w}\mathrm{op2}_1 = {}^{1:u:v}_{1}\mathrm{op2}_1 + {}^{1:u:v}c_{od}\mathrm{op2}_1 \bullet C_{w} + {}^{1:u:v}c_{ev}\mathrm{op2}_1 \bullet D_{w}$

etc.

*1 $\quad {}^{1}_{2}c_{od}\mathrm{op2}_1 = {}^{1}_{2}\mathrm{op2}_1 - {}^{1}_{1}\mathrm{op2}_1 = 1 - 41 = -40$

*2 $\quad {}^{1}_{2}c_{ev}\mathrm{op2}_1 = {}^{1}\mathrm{op1}_1 - {}^{1}_{2}c_{od}\mathrm{op2}_1 = 41 - (-40) = 81$

*3 $\quad {}^{1:v}c_{od}\mathrm{op2}_1 = {}^{1:v}_{2}\mathrm{op2}_1 - {}^{1:v}_{1}\mathrm{op2}_1$

*4 $\quad {}^{1:v}c_{ev}\mathrm{op2}_1 = {}^{1:v}\mathrm{op1}_1 - {}^{1:v}c_{od}\mathrm{op2}_1$

*5 $\quad {}^{1:u:v}c_{od}\mathrm{op2}_1 = {}^{1:u:v}_{2}\mathrm{op2}_1 - {}^{1:u:v}_{1}\mathrm{op2}_1$

*6 $\quad {}^{1:u:v}c_{ev}\mathrm{op2}_1 = {}^{1:u:v}\mathrm{op1}_1 - {}^{1:u:v}c_{od}\mathrm{op2}_1$

Chart 71. The general method for deriving any $\mathrm{op1}_1/\mathrm{op2}_1$ pair.

Predicting Any Factor Pair

All Ingredients Present

We now have all of the ingredients that we need in order to predict the factor pair at any location in the plane-domain that we have been exploring all of this time. The combination of the general methods for finding z, ±, and $op1_1/op2_1$ pairs is sufficient for our task.

The Methods in Action

Let us show these general methods working in concert on an actual problem, to fully illustrate their power and functioning. Our test will be to find the first few factor pairs for each of the first five series in column ${}^{1:5:5:6:6}S$.

We will test the overall method against the data that we already have for these series (refer to Tables 97 and 100).

Calculating the First Three Factor Pairs for the First Five Series in Family ${}^{1:5:5:6:6}S$

z's

Let us begin by calculating z for the five target series:

$${}_1z = 1$$

$$\begin{aligned} {}^{1}_{5}z &= G_5 + {}_1zA_5 \\ &= 1 + 1 \cdot 2 \\ &= 1 + 2 \\ &= 3 \end{aligned}$$

$$\begin{aligned} {}^{1:5}_{5}z &= {}_1z \cdot G_5 + {}^{1}_{5}z \cdot A_5 \\ &= 1 \cdot 1 + 3 \cdot 2 \\ &= 1 + 6 \\ &= 7 \end{aligned}$$

$$
\begin{aligned}
{}^{1:5:5}{}_{6}z &= {}^{1}{}_{5}zG_{6} + {}^{1:5}{}_{5}zA_{6} \\
&= 3 \bullet -1 + 7 \bullet 3 \\
&= -3 + 21 \\
&= 18
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:5:5:6}{}_{6}z &= {}^{1:5}{}_{5}zG_{6} + {}^{1:5:5}{}_{6}zA_{6} \\
&= 7 \bullet -1 + 18 \bullet 3 \\
&= -7 + 54 \\
&= 47
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:5:5:6:6}{}_{1}z &= {}^{1:5:5}{}_{6}zG_{1} + {}^{1:5:5:6}{}_{6}zA_{1} \\
&= 18 \bullet 1 + 47 \bullet 0 \\
&= 18 + 0 \\
&= 18
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:5:5:6:6}{}_{2}z &= {}^{1:5:5}{}_{6}zG_{2} + {}^{1:5:5:6}{}_{6}zA_{2} \\
&= 18 \bullet -1 + 47 \bullet 1 \\
&= -18 + 47 \\
&= 29
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:5:5:6:6}{}_{3}z &= {}^{1:5:5}{}_{6}zG_{3} + {}^{1:5:5:6}{}_{6}zA_{3} \\
&= 18 \bullet 1 + 47 \bullet 1 \\
&= 18 + 47 \\
&= 65
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:5:5:6:6}{}_{4}z &= {}^{1:5:5}{}_{6}zG_{4} + {}^{1:5:5:6}{}_{6}zA_{4} \\
&= 18 \bullet -1 + 47 \bullet 2 \\
&= -18 + 94 \\
&= 76
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:5:5:6:6}{}_{5}z &= {}^{1:5:5}{}_{6}zG_{5} + {}^{1:5:5:6}{}_{6}zA_{5} \\
&= 18 \bullet 1 + 47 \bullet 2 \\
&= 18 + 94 \\
&= 112
\end{aligned}
$$

Next we will calculate the ± values for these five series. Before we do, though, we will make some observations about z.

Some Observations on z

When We Really Need z Now

When the ± calculation method was first developed, inspired by the then recent success in finding the general method for finding z at any depth, the ± calculations were pursued by using z - ± as A_w's coefficient rather than using ± alt. By using ± alt instead, which is now better understood, there is no need to involve z in the ± calculations. The only time that z is needed in these general methods, it therefore turns out, is when converting a ± value to its corresponding op2 spacing values.

Do We Still Need the General Method for z?

Do we still need to use the general method for finding z's, if only to help with ±-to-op2 spacing conversions? No. Recall that when we calculate a ± value, the pair of values for ± and ± alt add up to z. So, if we find ± and ± alt for a series, then we have also found z. Therefore, we actually do not need to use the general method for z any longer.

±'s

Continuing on, then, let us calculate the ± values for the five series in our test example:

$${}_1\pm_1 = (-0.5, 1.5)$$

$${}_1\pm = (-0.5, 1.5)$$

$$\begin{aligned} {}^{1}{}_{5}\pm &= {}_1\pm_1 \cdot G_5 + {}_1\pm \text{ alt} \cdot A_5 \\ &= (-0.5, 1.5) \cdot 1 + (1.5, -0.5) \cdot 2 \\ &= (-0.5, 1.5) + (3.0, -1.0) \\ &= (2.5, 0.5) \end{aligned}$$

$$\begin{aligned} {}^{1:5}{}_{5}\pm &= {}_1\pm \cdot G_5 + {}^{1}{}_{5}\pm \text{ alt} \cdot A_5 \\ &= (-0.5, 1.5) \cdot 1 + (0.5, 2.5) \cdot 2 \\ &= (-0.5, 1.5) + (1.0, 5.0) \\ &= (0.5, 6.5) \end{aligned}$$

$$\begin{aligned} {}^{1:5:5}{}_{6}\pm &= {}^{1}{}_{5}\pm \cdot G_6 + {}^{1:5}{}_{5}\pm \text{ alt} \cdot A_6 \\ &= (2.5, 0.5) \cdot -1 + (6.5, 0.5) \cdot 3 \\ &= (-2.5, -0.5) + (19.5, 1.5) \\ &= (17.0, 1.0) \end{aligned}$$

$$\begin{aligned} {}^{1:5:5:6}{}_{6}\pm &= {}^{1:5}{}_{5}\pm \cdot G_6 + {}^{1:5:5}{}_{6}\pm \text{ alt} \cdot A_6 \\ &= (0.5, 6.5) \cdot -1 + (1.0, 17.0) \cdot 3 \\ &= (-0.5, -6.5) + (3.0, 51.0) \\ &= (2.5, 44.5) \end{aligned}$$

$$\begin{aligned} {}^{1:5:5:6:6}{}_{1}\pm &= {}^{1:5:5}{}_{6}\pm \cdot G_1 + {}^{1:5:5:6}{}_{6}\pm \text{ alt} \cdot A_1 \\ &= (17.0, 1.0) \cdot 1 + (44.5, 2.5) \cdot 0 \\ &= (17.0, 1.0) + 0 \\ &= (17.0, 1.0) \end{aligned}$$

$$
\begin{aligned}
{}^{1:5:5:6:6}{}_{2}\pm &= {}^{1:5:5}{}_{6}\pm \cdot G_2 + {}^{1:5:5:6}{}_{6}\pm\ \text{alt}\cdot A_2 \\
&= (17.0,1.0)\cdot -1 + (44.5,2.5)\cdot 1 \\
&= (-17.0,-1.0) + (44.5,2.5) \\
&= (27.5,1.5)
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:5:5:6:6}{}_{3}\pm &= {}^{1:5:5}{}_{6}\pm \cdot G_3 + {}^{1:5:5:6}{}_{6}\pm\ \text{alt}\cdot A_3 \\
&= (17.0,1.0)\cdot 1 + (44.5,2.5)\cdot 1 \\
&= (17.0,1.0) + (44.5,2.5) \\
&= (61.5,3.5)
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:5:5:6:6}{}_{4}\pm &= {}^{1:5:5}{}_{6}\pm \cdot G_4 + {}^{1:5:5:6}{}_{6}\pm\ \text{alt}\cdot A_4 \\
&= (17.0,1.0)\cdot -1 + (44.5,2.5)\cdot 2 \\
&= (-17.0,-1.0) + (89.0,5.0) \\
&= (72.0,4.0)
\end{aligned}
$$

$$
\begin{aligned}
{}^{1:5:5:6:6}{}_{5}\pm &= {}^{1:5:5}{}_{6}\pm \cdot G_5 + {}^{1:5:5:6}{}_{6}\pm\ \text{alt}\cdot A_5 \\
&= (17.0,1.0)\cdot 1 + (44.5,2.5)\cdot 2 \\
&= (17.0,1.0) + (89.0,5.0) \\
&= (106.0,6.0)
\end{aligned}
$$

Reconfirming That These ± Plus ± alt Sums Equal the z's

Before we convert these ± values to op2 spacings, let us reconfirm that the sum of these ± and ± alt pairs is equal to the z for the given series:

Series	z	±Pair	± Pair Sum
${}^{1:5:5:6:6}{}_{1}S$	18	(17.0,1.0)	18.0
${}^{1:5:5:6:6}{}_{2}S$	29	(27.5,1.5)	29.0
${}^{1:5:5:6:6}{}_{3}S$	65	(61.5,3.5)	65.0
${}^{1:5:5:6:6}{}_{4}S$	76	(72.0,4.0)	76.0
${}^{1:5:5:6:6}{}_{5}S$	112	(106.0,6.0)	112.0

Table 151. z vs. sum of ± and ± alt for series ${}^{1:5:5:6:6}{}_{1}S$ through ${}^{1:5:5:6:6}{}_{5}S$.

±-to-op2 Spacing Conversions

The conversions of these ± values to op2 spacing coefficients are:

$$
\begin{aligned}
{}^{1:5:5:6:6}{}_{1}S \text{ op2 spacing} &= (z^2 \pm 2z\cdot\text{"}\pm\text{"})\div 2 \\
&= [18^2 \pm 2\cdot 18\cdot(17.0 \text{ or } 1.0)]\div 2 \\
&= [324 \pm 36\cdot(17.0 \text{ or } 1.0)]\div 2 \\
&= [324 \pm (612 \text{ or } 36)]\div 2 \\
&= [(324 \pm 612) \text{ or } (324 \pm 36)]\div 2 \\
&= [(936,-288) \text{ or } (360,288)]\div 2 \\
&= (468,-144) \text{ or } (180,144) \\
&\rightarrow (468\cdot,-144\cdot) \text{ or } (180\cdot,144\cdot)
\end{aligned}
$$

$^{1:5:5:6:6}{}_{2}S$ op2 spacing $= (z^2 \pm 2z\cdot"\pm")\div 2$
$= [29^2 \pm 2\cdot 29\cdot(27.5 \text{ or } 1.5)]\div 2$
$= [841 \pm 58\cdot(27.5 \text{ or } 1.5)]\div 2$
$= [841 \pm (1595 \text{ or } 87)]\div 2$
$= [(841 \pm 1595) \text{ or } (841 \pm 87)]\div 2$
$= [(2436,-754) \text{ or } (928,754)]\div 2$
$= (1218,-377) \text{ or } (464,377)$
$\rightarrow (1218\cdot,-377\cdot) \text{ or } (464\cdot,377\cdot)$

$^{1:5:5:6:6}{}_{3}S$ op2 spacing $= (z^2 \pm 2z\cdot"\pm")\div 2$
$= [65^2 \pm 2\cdot 65\cdot(61.5 \text{ or } 3.5)]\div 2$
$= [4225 \pm 130\cdot(61.5 \text{ or } 3.5)]\div 2$
$= [4225 \pm (7995 \text{ or } 455)]\div 2$
$= [(4225 \pm 7995) \text{ or } (4225 \pm 455)]\div 2$
$= [(12220,-3770) \text{ or } (4680,3770)]\div 2$
$= (6110,-1885) \text{ or } (2340,1885)$
$\rightarrow (6110\cdot,-1885\cdot) \text{ or } (2340\cdot,1885\cdot)$

$^{1:5:5:6:6}{}_{4}S$ op2 spacing $= (z^2 \pm 2z\cdot"\pm")\div 2$
$= [76^2 \pm 2\cdot 76\cdot(72.0 \text{ or } 4.0)]\div 2$
$= [5776 \pm 152\cdot(72.0 \text{ or } 4.0)]\div 2$
$= [5776 \pm (10944 \text{ or } 608)]\div 2$
$= [(5776 \pm 10944) \text{ or } (5776 \pm 608)]\div 2$
$= [(16720,-5168) \text{ or } (6384,5168)]\div 2$
$= (8360,-2584) \text{ or } (3192,2584)$
$\rightarrow (8360\cdot,-2584\cdot) \text{ or } (3192\cdot,2584\cdot)$

$^{1:5:5:6:6}{}_{5}S$ op2 spacing $= (z^2 \pm 2z\cdot"\pm")\div 2$
$= [112^2 \pm 2\cdot 112\cdot(106.0 \text{ or } 6.0)]\div 2$
$= [12544 \pm 224\cdot(106.0 \text{ or } 6.0)]\div 2$
$= [12544 \pm (23744 \text{ or } 1344)]\div 2$
$= [(12544 \pm 23744) \text{ or } (12544 \pm 1344)]\div 2$
$= [(36288,-11200) \text{ or } (13888,11200)]\div 2$
$= (18144,-5600) \text{ or } (6944,5600)$
$\rightarrow (18144\cdot,-5600\cdot) \text{ or } (6944\cdot,5600\cdot)$

Starting Factor Pairs for the Five Series

Let us now calculate the starting factor pairs for the five test series:

$^{1}{}_{1}op2_{1} = 41$

$^{1}{}_{2}op2_{1} = 1$

$^{1}op1_{1} = 41$

$^{1}{}_{5\text{-}2}op2_{1} = {}^{1}{}_{1}op2_{1} + {}^{1}c_{od}op2_{1}\cdot C_{5\text{-}2} + {}^{1}c_{ev}op2_{1}\cdot D_{5\text{-}2}$

$^{1}c_{od}op2_1 = {}^{1}{}_{2}op2_1 - {}^{1}{}_{1}op2_1$
$= 1 - 41$
$= -40$

$^{1}c_{ev}op2_1 = {}^{1}op1_1 - {}^{1}c_{od}op2_1$
$= 41 - -40$
$= 81$

$^{1}{}_{5-2}op2_1 = 41 + -40 \cdot C_{5-2} + 81 \cdot D_{5-2}$
$= 41 - 40C_3 + 81D_3$

$C_3 = A_3^{\,2}$
$= 1^2$
$= 1$

$D_3 = B_3(B_3+1)$
$= 1 \cdot 2$
$= 2$

$^{1}{}_{3}op2_1 = 41 - 40 \cdot 1 + 81 \cdot 2$
$= 41 - 40 + 162$
$= 163$

$^{1}{}_{5}op2_1 = {}^{1}{}_{1}op2_1 + {}^{1}c_{od}op2_1 \cdot C_5 + {}^{1}c_{ev}op2_1 \cdot D_5$
$= 41 + -40 \cdot C_5 + 81 \cdot D_5$

$C_5 = A_5^{\,2}$
$= 2^2$
$= 4$

$D_5 = B_5(B_5+1)$
$= 2 \cdot 3$
$= 6$

$^{1}{}_{5}op2_1 = 41 + -40 \cdot 4 + 81 \cdot 6$
$= 41 - 160 + 486$
$= 367$

****

$^{1:5}{}_{1}op2_1 = {}^{1}op1_1$
$= 41$

$^{1:5}{}_{2}op2_1 = {}^{1}{}_{5-2}op2_1$
$= 163$

$^{1:5}op1_1 = {}^{1}{}_{5}op2_1$
$= 367$

$^{1:5}{}_{5-2}op2_1 = {}^{1:5}{}_{1}op2_1 + {}^{1:5}c_{od}op2_1 \cdot C_{5-2} + {}^{1:5}c_{ev}op2_1 \cdot D_{5-2}$

$$ {}^{1:5}c_{od}op2_1 = {}^{1:5}{}_{2}op2_1 - {}^{1:5}{}_{1}op2_1 $$
$$ = 163 - 41 $$
$$ = 122 $$

$$ {}^{1:5}c_{ev}op2_1 = {}^{1:5}op1_1 - {}^{1:5}c_{od}op2_1 $$
$$ = 367 - 122 $$
$$ = 245 $$

$C_3 = 1$

$D_3 = 2$

$$ {}^{1:5}{}_{3}op2_1 = 41 + 122 \bullet C_3 + 245 \bullet D_3 $$
$$ = 41 + 122 \bullet 1 + 245 \bullet 2 $$
$$ = 41 + 122 + 490 $$
$$ = 653 $$

$C_5 = 4$

$D_5 = 6$

$$ {}^{1:5}{}_{5}op2_1 = {}^{1:5}{}_{1}op2_1 + {}^{1:5}c_{od}op2_1 \bullet C_5 + {}^{1:5}c_{ev}op2_1 \bullet D_5 $$
$$ = 41 + 122 \bullet C_5 + 245 \bullet D_5 $$
$$ = 41 + 122 \bullet 4 + 245 \bullet 6 $$
$$ = 41 + 488 + 1470 $$
$$ = 1999 $$

****

$$ {}^{1:5:5}{}_{1}op2_1 = {}^{1:5}op1_1 $$
$$ = 367 $$

$$ {}^{1:5:5}{}_{2}op2_1 = {}^{1:5}{}_{3}op2_1 $$
$$ = 653 $$

$$ {}^{1:5:5}op1_1 = {}^{1:5}{}_{5}op2_1 $$
$$ = 1999 $$

$$ {}^{1:5:5}{}_{6-2}op2_1 = {}^{1:5:5}{}_{1}op2_1 + {}^{1:5:5}c_{od}op2_1 \bullet C_{6-2} + {}^{1:5:5}c_{ev}op2_1 \bullet D_{6-2} $$

$$ {}^{1:5:5}c_{od}op2_1 = {}^{1:5:5}{}_{2}op2_1 - {}^{1:5:5}{}_{1}op2_1 $$
$$ = 653 - 367 $$
$$ = 286 $$

$$ {}^{1:5:5}c_{ev}op2_1 = {}^{1:5:5}op1_1 - {}^{1:5:5}c_{od}op2_1 $$
$$ = 1999 - 286 $$
$$ = 1713 $$

$$C_{6-2} = C_4 = A_4^{\,2} = 2^2 = 4$$

$$D_{6-2} = D_4 = B_4(B_4+1) = 1 \cdot 2 = 2$$

$$^{1:5:5}{}_4op2_1 = 367 + 286 \cdot 4 + 1713 \cdot 2 = 367 + 1144 + 3426 = 4937$$

$$^{1:5:5}{}_6op2_1 = {}^{1:5:5}{}_1op2_1 + {}^{1:5:5}C_{od}op2_1 \cdot C_6 + {}^{1:5:5}C_{ev}op2_1 \cdot D_6$$

$$C_6 = A_6^{\,2} = 3^2 = 9$$

$$D_6 = B_6(B_6+1) = 2 \cdot 3 = 6$$

$$^{1:5:5}{}_6op2_1 = 367 + 286 \cdot 9 + 1713 \cdot 6 = 367 + 2574 + 10278 = 13219$$

****

$$^{1:5:5:6}{}_1op2_1 = {}^{1:5:5}op1_1 = 1999$$

$$^{1:5:5:6}{}_2op2_1 = {}^{1:5:5}{}_4op2_1 = 4937$$

$$^{1:5:5:6}op1_1 = {}^{1:5:5}{}_6op2_1 = 13219$$

$$^{1:5:5:6}{}_4op2_1 = {}^{1:5:5:6}{}_1op2_1 + {}^{1:5:5:6}C_{od}op2_1 \cdot C_4 + {}^{1:5:5:6}C_{ev}op2_1 \cdot D_4$$

$$^{1:5:5:6}C_{od}op2_1 = {}^{1:5:5:6}{}_2op2_1 - {}^{1:5:5:6}{}_1op2_1 = 4937 - 1999 = 2938$$

$$^{1:5:5:6}C_{ev}op2_1 = {}^{1:5:5:6}op1_1 - {}^{1:5:5:6}C_{od}op2_1 = 13219 - 2938 = 10281$$

$$C_4 = 4$$

$D_4 = 2$

$$^{1:5:5:6}{}_4op2_1 = 1999 + 2938 \cdot 4 + 10281 \cdot 2$$
$$= 1999 + 11752 + 20562$$
$$= 34313$$

$C_6 = 9$

$D_6 = 6$

$$^{1:5:5:6}{}_6op2_1 = 1999 + 2938 \cdot 9 + 10281 \cdot 6$$
$$= 1999 + 26442 + 61686$$
$$= 90127$$

****

$$^{1:5:5:6:6}{}_1op2_1 = {}^{1:5:5:6}op1_1$$
$$= 13219$$

$$^{1:5:5:6:6}{}_2op2_1 = {}^{1:5:5:6}{}_4op2_1$$
$$= 34313$$

$$^{1:5:5:6:6}op1_1 = {}^{1:5:5:6}{}_6op2_1$$
$$= 90127$$

$$^{1:5:5:6:6}{}_2op2_1 = {}^{1:5:5:6:6}{}_1op2_1 + {}^{1:5:5:6:6}c_{od}op2_1 \cdot C_2 + {}^{1:5:5:6:6}c_{ev}op2_1 \cdot D_2$$

(let's make believe that we don't already know this item)

$$^{1:5:5:6:6}c_{od}op2_1 = {}^{1:5:5:6:6}{}_2op2_1 - {}^{1:5:5:6:6}{}_1op2_1$$
$$= 34313 - 13219$$
$$= 21094$$

$$^{1:5:5:6:6}c_{ev}op2_1 = {}^{1:5:5:6:6}op1_1 - {}^{1:5:5:6:6}c_{od}op2_1$$
$$= 90127 - 21094$$
$$= 69033$$

$$C_2 = A_2^2$$
$$= 1^2$$
$$= 1$$

$$D_2 = B_2(B_2+1)$$
$$= 0 \cdot 1$$
$$= 0$$

$$^{1:5:5:6:6}{}_2op2_1 = 13219 + 21094 \cdot 1 + 69033 \cdot 0$$
$$= 13219 + 21094 + 0$$
$$= 34313$$

$$^{1:5:5:6:6}{}_3op2_1 = {}^{1:5:5:6:6}{}_1op2_1 + {}^{1:5:5:6:6}c_{od}op2_1 \cdot C_3 + {}^{1:5:5:6:6}c_{ev}op2_1 \cdot D_3$$

$C_3 = 1$

$D_3 = 2$

$$^{1:5:5:6:6}{}_{3}op2_1 = 13219 + 21094 \cdot 1 + 69033 \cdot 2$$
$$= 13219 + 21094 + 138066$$
$$= 172379$$

$$^{1:5:5:6:6}{}_{4}op2_1 = {}^{1:5:5:6:6}{}_{1}op2_1 + {}^{1:5:5:6:6}c_{od}op2_1 \cdot C_4 + {}^{1:5:5:6:6}c_{ev}op2_1 \cdot D_4$$

$C_4 = 4$

$D_4 = 2$

$$^{1:5:5:6:6}{}_{4}op2_1 = 13219 + 21094 \cdot 4 + 69033 \cdot 2$$
$$= 13219 + 84376 + 138066$$
$$= 235661$$

$$^{1:5:5:6:6}{}_{5}op2_1 = {}^{1:5:5:6:6}{}_{1}op2_1 + {}^{1:5:5:6:6}c_{od}op2_1 \cdot C_5 + {}^{1:5:5:6:6}c_{ev}op2_1 \cdot D_5$$

$C_5 = 4$

$D_5 = 6$

$$^{1:5:5:6:6}{}_{5}op2_1 = 13219 + 21094 \cdot 4 + 69033 \cdot 6$$
$$= 13219 + 84376 + 414198$$
$$= 511793$$

Finding the op1 Spacing

There is just one more ingredient that we need to come up with in order to finish our example. We need the op1 spacing for the family of series. Obtaining this is almost trivial. It is simply the spacing for the op1 values when they were the parent op2 values. The $op1_1$ value for family $^{1:5:5:6:6}S$ is 90127. It was the $op2_1$ value for series $^{1:5:5:6}{}_{6}S$. That series's op2 spacing is a function of its ± value. This could be computed directly or, if the ± calculations have already been done, picked right from them.

The op1 Spacings

Since we have done the ± calculations, let us just extract $^{1:5:5:6}{}_{6}S$'s ± value from them: (2.5,44.5). We use the same ±-to-op2 spacing conversion process here, even though we are converting to op1 spacings. One note of importance: we use the alternate ± to line up the proper op1 and op2 spacings in the target series. z = 2.5 + 44.5 = 47.0.

$$^{1...t:u:v}\ op1\ spacing = (z^2 \pm 2z \cdot {}^{1...u}{}_{v} \pm alt) \div 2$$

$^{1:5:5:6:6}S$ op1 spacing $= [47^2 \pm 2 \cdot 47 \cdot (2.5 \text{ or } 44.5)] \div 2$
$= [2209 \pm 94 \cdot (2.5 \text{ or } 44.5)] \div 2$
$= [2209 \pm (235 \text{ or } 4183)] \div 2$
$= [(2209 \pm 235) \text{ or } (2209 \pm 4183)] \div 2$
$= [(2444,1974) \text{ or } (6392,-1974)] \div 2$
$= (1222,987) \text{ or } (3196,-987)$
$\rightarrow (1222\bullet,987\bullet) \text{ or } (3196\bullet,-987\bullet)$

Result Summary

Summarizing the results of our calculations for the five $^{1:5:5:6:6}S$ series, we have found that

$op1_1$ for all five series is 90127

the $op2_1$'s are 13219, 34313, 172379, 235661, and 511793

the op1 spacing is 3196•,-987• or 1222•,987•

and the op2 spacings are:

468•,-144• or 180•,144•

1218•,-377• or 464•,377•

6110•,-1885• or 2340•,1885•

8360•,-2584• or 3192•,2584•

18144•,-5600• or 6944•,5600•.

Our Calculations Work

By referring now to Tables 97 and 100 we can readily see that our $op1_1$ and $op2_1$ values match perfectly those derived earlier. We can also see that the op1 spacings and op2 spacings in Tables 97 and 100 match the second choices in our "or's" above.

Finding Any Factor Pair

So, we see that our general calculation methods do indeed work. Where do we stand, then?

We have general calculation methods for generating any factor pair desired:

$$^{1...v}{}_{w}op1_{x} = {}^{1...v}{}_{w}op1_{1} + {}^{1...v}{}_{w}c_{od}op1 \cdot A_{x} + {}^{1...v}{}_{w}c_{ev}op1 \cdot B_{x}$$
$$^{1...v}{}_{w}op2_{x} = {}^{1...v}{}_{w}op2_{1} + {}^{1...v}{}_{w}c_{od}op2 \cdot A_{x} + {}^{1...v}{}_{w}c_{ev}op2 \cdot B_{x}.$$

Find the series starting factor pair, ${}^{1...v}{}_{w}op1_{1}$ and ${}^{1...v}{}_{w}op2_{1}$, using the method of Chart 71 (see page 149).

Find the op2 spacing coefficients for the series, ${}^{1...v}{}_{w}c_{od}op2$ and ${}^{1...v}{}_{w}c_{ev}op2$, using the method for arriving at the ± values detailed under "Re-expressing the Method as a Cascade" on page 144. Then convert the resulting ± value to op2 spacing as described in "Converting ± to op2 Spacing" on page 140 (remember that z is equal to the sum of the two alternative ± values found -- this will be useful when using the conversion process).

Finally, the op1 spacing values for the series, ${}^{1...v}{}_{w}c_{od}op1$ and ${}^{1...v}{}_{w}c_{ev}op1$, are available from the ± calculations just done -- the op1 spacings are the same as the op2 spacings for the *parent* series (see "op1 Spacing from Parent op2 Spacing" on page 97).

Open Questions

Certain questions remain. Are *all* composite outputs of $x^2 + x + 41$ accounted for by this "plane-world" of factor pairs? Is there any way to leverage this knowledge to tell if a given output of $x^2 + x + 41$ is prime or not? Can this same methodology be used to investigate other $x^2 + x + c$ output series, each such formula possibly having its own "plane-world"? Finally, what insight can this work contribute to our understanding of the onset of chaos? Only time and further work will help us tell what the answers to these questions are.

Appendix A -- Family Parameter Charts

	^1_1S	^1_2S	^1_3S	^1_4S	^1_5S	^1_6S
x_1	40	81	122	163	204	245
operands	41•41	41•163	41•367	41•653	41•1021	41•1471
x spacing	1•1,3,5... 0•2,4,6... 1	1• 1• 2	1• 2• 3	1• 3• 4	1• 4• 5	1• 5• 6
op2 spacing	2•1,3,5... -1•2,4,6... 1	4• 0• 4	6• 3• 9	8• 8• 16	10• 15• 25	12• 24• 36
$op2_1$:41	1•41-0	4•41-1	9•41-2	16•41-3	25•41-4	36•41-5
x_1:41	1•41-1	2•41-1	3•41-1	4•41-1	5•41-1	6•41-1
op1 spacing	0•1,3,5... 1•2,4,6... 1					

Chart 1. Family ^1_wS.

	^2_1S	^2_2S	^2_3S	^2_4S	^2_5S
x_1	244	407	570	733	896
operands	163•367	163•1019	163•1997	163•3301	163•4931
x spacing	5•1,3,5... 1•2,4,6... 6	9• 1• 10	13• 1• 14	17• 1• 18	21• 1• 22
op2 spacing	6•1,3,5... 3•2,4,6... 9	20• 5• 25	42• 7• 49	72• 9• 81	110• 11• 121
$op2_1$:41	9•41-2	25•41-6	49•41-12	81•41-20	121•41-30
x_1:41	6•41-2	10•41-3	14•41-4	18•41-5	22•41-6
op1 spacing	4•1,3,5... 0•2,4,6... 4				

Chart 2. Family ^3_wS (originally ^2_wS).

	${}^{3}_{1}S$	${}^{3}_{2}S$	${}^{3}_{3}S$	${}^{3}_{4}S$	${}^{3}_{5}S$
x_1	244	489	611	856	978
operands	367•163	367•653	367•1019	367•1999	367•2609
x spacing	5•1,3,5... 1•2,4,6... 6	7• 5• 12	11• 4• 15	13• 8• 21	17• 7• 24
op2 spacing	4•1,3,5... 0•2,4,6... 4	8• 8• 16	20• 5• 25	28• 21• 49	48• 16• 64
$op2_1$:41	4•41-1	16•41-3	25•41-6	49•41-10	64•41-15
x_1:41	6•41-2	12•41-3	15•41-4	21•41-5	24•41-6
op1 spacing	6•1,3,5... 3•2,4,6... 9				

Chart 3. Family ${}^{5}_{w}S$ (originally ${}^{3}_{w}S$).

	${}^{4}_{1}S$	${}^{4}_{2}S$	${}^{4}_{3}S$
x_1	163	489	816
operands	653•41	653•367	653•1021
x spacing	1•1,3,5... 3•2,4,6... 4	7• 5• 12	9• 11• 20
op2 spacing	0•1,3,5... 1•2,4,6... 1	6• 3• 9	10• 15• 25
$op2_1$:41	1•41-0	9•41-2	25•41-4
x_1:41	4•41-1	12•41-3	20•41-4
op1 spacing	8•1,3,5... 8•2,4,6... 16		

Chart 4. Family ${}^{7}_{w}S$ (originally ${}^{4}_{w}S$).

	${}^{4}_{1}S$	${}^{4}_{2}S$	${}^{4}_{3}S$	${}^{4}_{4}S$	${}^{4}_{5}S$	${}^{4}_{6}S$
x_1	163	489	816	1142	1469	1795
operands	653•41	653•367	653•1021	653•1999	653•3307	653•4937
x spacing	1•1,3,5... 3•2,4,6... 4	7• 5• 12	9• 11• 20	15• 13• 28	17• 19• 36	23• 21• 44
op2 spacing	0•1,3,5... 1•2,4,6... 1	6• 3• 9	10• 15• 25	28• 21• 49	36• 45• 81	66• 55• 121
$op2_1$:41	1•41-0	9•41-2	25•41-4	49•41-10	81•41-14	121•41-24
x_1:41	4•41-1	12•41-3	20•41-4	28•41-6	36•41-7	44•41-9
op1 spacing	8•1,3,5... 8•2,4,6... 16					

Chart 5. Family ${}^{7}_{w}S$ (originally ${}^{4}_{w}S$).

	${}^{5}_{1}S$	${}^{5}_{2}S$	${}^{5}_{3}S$	${}^{5}_{4}S$	${}^{5}_{5}S$	${}^{5}_{6}S$
x_1	204	816	1225	1837	2246	2858
operands	1021•41	1021•653	1021•1471	1021•3307	1021•4943	1021•8003
x spacing	1•1,3,5... 4•2,4,6... 5	9• 11• 20	11• 19• 30	19• 26• 45	21• 34• 55	29• 41• 70
op2 spacing	0•1,3,5... 1•2,4,6... 1	8• 8• 16	12• 24• 36	36• 45• 81	44• 77• 121	84• 112• 196
$op2_1$:41	1•41-0	16•41-3	36•41-5	81•41-14	121•41-18	196•41-33
x_1:41	5•41-1	20•41-4	30•41-5	45•41-8	55•41-9	70•41-12
op1 spacing	10•1,3,5... 15•2,4,6... 25					

Chart 6. Family ${}^{9}_{w}S$ (originally ${}^{5}_{w}S$).

	$^1{}_1S$	$^1{}_2S$	$^1{}_3S$	$^1{}_4S$	$^1{}_5S$	$^1{}_6S$
x_1	40	81	122	163	204	245
operands	41•41	41•163	41•367	41•653	41•1021	41•1471
x spacing	1•1,3,5... 0•2,4,6... 1	3• -1• 2	5• -2• 3	7• -3• 4	9• -4• 5	11• -5• 6
op2 spacing	0•1,3,5... 1•2,4,6... 1	4• 0• 4	12• -3• 9	24• -8• 16	40• -15• 25	60• -24• 36
$op2_1$:41	1•41-0	4•41-1	9•41-2	16•41-3	25•41-4	36•41-5
x_1:41	1•41-1	2•41-1	3•41-1	4•41-1	5•41-1	6•41-1
op1 spacing	2•1,3,5... -1•2,4,6... 1					

Chart 7. Family $^1{}_wS$.

	$^1{}_1S$	$^1{}_2S$	$^1{}_3S$	$^1{}_4S$	$^1{}_5S$	$^1{}_6S$
x_1	40	0	81	41	122	82
operands	41•41	41•1	41•163	41•43	41•367	41•167
x spacing	1•1,3,5... 0•2,4,6... 1	-1• 1• 0	1• 1• 2	-1• 2• 1	1• 2• 3	-1• 3• 2
op2 spacing	2•1,3,5... -1•2,4,6... 1	0• 0• 0	4• 0• 4	-2• 3• 1	6• 3• 9	-4• 8• 4
$op2_1$:41	1•41-0	0•41+1	4•41-1	1•41+2	9•41-2	4•41+3
x_1:41	1•41-1	0•41-0	2•41-1	1•41-0	3•41-1	2•41-0
op1 spacing	0•1,3,5... 1•2,4,6... 1					

Chart 8. Family $^1{}_wS$.

	${}^{2}_{1}S$	${}^{2}_{2}S$	${}^{2}_{3}S$	${}^{2}_{4}S$	${}^{2}_{5}S$	${}^{2}_{6}S$
x_1	81	81	244	244	407	407
operands	163•41	163•41	163•367	163•367	163•1019	163•1019
x spacing	1•1,3,5... $\frac{1}{2}$•2,4,6...	3• $\frac{-1}{2}$•	5• $\frac{1}{6}$•	7• $\frac{-1}{6}$•	9• $\frac{1}{10}$•	11• $\frac{-1}{10}$•
op2 spacing	0•1,3,5... $\frac{1}{1}$•2,4,6...	2• $\frac{-1}{1}$•	6• $\frac{3}{9}$•	12• $\frac{-3}{9}$•	20• $\frac{5}{25}$•	30• $\frac{-5}{25}$•
$op2_1$:41	1•41-0	1•41-0	9•41-2	9•41-2	25•41-6	25•41-6
x_1:41	2•41-1	2•41-1	6•41-2	6•41-2	10•41-3	10•41-3
op1 spacing	4•1,3,5... $\frac{0}{4}$•2,4,6...					

Chart 9. Family ${}^{3}_{W}S$ (originally ${}^{2}_{W}S$).

x_1	407	611
operands	1019•163	1019•367
x spacing	9•1,3,5... $\frac{1}{10}$•2,4,6...	11• $\frac{4}{15}$•
op2 spacing	4•1,3,5... $\frac{0}{4}$•2,4,6...	6• $\frac{3}{9}$•
$op2_1$:41	4•41-1	9•41-2
x_1:41	10•41-3	15•41-4
op1 spacing	20•1,3,5... $\frac{5}{25}$•2,4,6...	

Chart 10. Two series in a prospective 1019 family.

x_1	407	611	1426	1630	2445
operands	1019•163	1019•367	1019•1997	1019•2609	1019•5869
x spacing	9•1,3,5... 1•2,4,6... 10	11• 4• 15	29• 6• 35	31• 9• 40	49• 11• 60
op2 spacing	4•1,3,5... 0•2,4,6... 4	6• 3• 9	42• 7• 49	48• 16• 64	120• 24• 144
$op2_1$:41	4•41-1	9•41-2	49•41-12	64•41-15	144•41-35
x_1:41	10•41-3	15•41-4	35•41-9	40•41-10	60•41-15
op1 spacing	20•1,3,5... 5•2,4,6... 25				

Chart 11. The 1019 family.

	$^{1:2}{}_{1}S$	$^{1:2}{}_{2}S$	$^{1:2}{}_{3}S$	$^{1:2}{}_{4}S$	$^{1:2}{}_{5}S$	$^{1:2}{}_{6}S$
x_1	0	0	1	1	2	2
operands	1•41	1•41	1•43	1•43	1•47	1•47
x spacing	1•1,3,5... -1•2,4,6... 0	-1• 1• 0	1• -1• 0	-1• 1• 0	1• -1• 0	-1• 1• 0
op2 spacing	2•1,3,5... -1•2,4,6... 1	0• 1• 1	4• -3• 1	-2• 3• 1	6• -5• 1	-4• 5• 1
$op2_1$:41	1•41-0	1•41-0	1•41+2	1•41+2	1•41+6	1•41+6
x_1:41	0•41-0	0•41-0	0•41+1	0•41+1	0•41+2	0•41+2
op1 spacing	0•1,3,5... 0•2,4,6... 0					
z	1	-1	1	-1	1	-1

Chart 12. Family $^{1:2}{}_{w}S$.

	1:1:1$_1$S	1:1:1$_2$S	1:1:1$_3$S
x_1	40	0	81
operands	41•41	41•1	41•163
x spacing	1•1,3,5... 0•2,4,6... 1	-1• 1• 0	1• 1• 2
op2 spacing	0•1,3,5... 1•2,4,6... 1	0• 0• 0	4• 0• 4
op2$_1$:41	1•41-0	0•41+1	4•41-1
x_1:41	1•41-1	0•41-0	2•41-1
op1 spacing	2•1,3,5... -1•2,4,6... 1		
z	1	0	2

Chart 13. Family $^{1:1:1}{}_wS$.

	1:1:2$_1$S	1:1:2$_2$S	1:1:2$_3$S
x_1	0	0	1
operands	1•41	1•41	1•43
x spacing	1•1,3,5... -1•2,4,6... 0	-1• 1• 0	1• -1• 0
op2 spacing	2•1,3,5... -1•2,4,6... 1	0• 1• 1	4• -3• 1
op2$_1$:41	1•41-0	1•41-0	1•41+2
x_1:41	0•41-0	0•41-0	0•41+1
op1 spacing	0•1,3,5... 0•2,4,6... 0		
z	1	-1	1

Chart 14. Family $^{1:1:2}{}_wS$.

	$^{1:1:3}{}_1S$	$^{1:1:3}{}_2S$	$^{1:1:3}{}_3S$
x_1	81	81	244
operands	163•41	163•41	163•367
x spacing	3•1,3,5... -1•2,4,6... 2	1• 1• 2	7• -1• 6
op2 spacing	2•1,3,5... -1•2,4,6... 1	0• 1• 1	12• -3• 9
$op2_1$:41	1•41-0	1•41-0	9•41-2
x_1:41	2•41-1	2•41-1	6•41-2
op1 spacing	4•1,3,5... 0•2,4,6... 4		
z	1	1	3

Chart 15. Family $^{1:1:3}{}_wS$.

	$^{1:1:4}{}_1S$	$^{1:1:4}{}_2S$	$^{1:1:4}{}_3S$
x_1	41	1	84
operands	43•41	43•1	43•167
x spacing	3•1,3,5... -2•2,4,6... 1	1• -1• 0	7• -5• 2
op2 spacing	2•1,3,5... -1•2,4,6... 1	0• 0• 0	12• -8• 4
$op2_1$:41	1•41-0	0•41+1	4•41+3
x_1:41	1•41-0	0•41+1	2•41+2
op1 spacing	4•1,3,5... -3•2,4,6... 1		
z	1	0	2

Chart 16. Family $^{1:1:4}{}_wS$.

	$1{:}1{:}5_{1}S$	$1{:}1{:}5_{2}S$	$1{:}1{:}5_{3}S$
x_1	122	244	489
operands	367•41	367•163	367•653
x spacing	5•1,3,5... -2•2,4,6... 3	7• -1• 6	17• -5• 12
op2 spacing	2•1,3,5... -1•2,4,6... 1	4• 0• 4	24• -8• 16
$op2_1$:41	1•41-0	4•41-1	16•41-3
x_1:41	3•41-1	6•41-2	12•41-3
op1 spacing	12•1,3,5... -3•2,4,6... 9		
z	1	2	4

Chart 17. Family $1^{:1:5}{}_{w}S$.

	$1{:}1{:}6_{1}S$	$1{:}1{:}6_{2}S$	$1{:}1{:}6_{3}S$
x_1	82	84	249
operands	167•41	167•43	167•373
x spacing	5•1,3,5... -3•2,4,6... 2	7• -5• 2	17• -11• 6
op2 spacing	2•1,3,5... -1•2,4,6... 1	4• -3• 1	24• -15• 9
$op2_1$:41	1•41-0	1•41+2	9•41+4
x_1:41	2•41-0	2•41+2	6•41+3
op1 spacing	12•1,3,5... -8•2,4,6... 4		
z	1	1	3

Chart 18. Family ${}^{1:1:6}{}_{w}S$.

	${}^{1:2:1}_{1}S$	${}^{1:2:1}_{2}S$	${}^{1:2:1}_{3}S$
x_1	0	40	41
operands	41•1	41•41	41•43
x spacing	-1•1,3,5... $\underline{1}$•2,4,6... 0	1• $\underline{0}$• 1	-1• $\underline{2}$• 1
op2 spacing	0•1,3,5... $\underline{0}$•2,4,6... 0	2• $\underline{-1}$• 1	-2• $\underline{3}$• 1
$op2_1$:41	0•41+1	1•41-0	1•41+2
x_1:41	0•41-0	1•41-1	1•41-0
op1 spacing	0•1,3,5... $\underline{1}$•2,4,6... 1		
z	0	1	1

Chart 19. Family ${}^{1:2:1}_{w}S$.

	${}^{1:2:2}_{1}S$	${}^{1:2:2}_{2}S$	${}^{1:2:2}_{3}S$
x_1	0	40	41
operands	41•1	41•41	41•43
x spacing	1•1,3,5... $\underline{-1}$•2,4,6... 0	1• $\underline{0}$• 1	3• $\underline{-2}$• 1
op2 spacing	0•1,3,5... $\underline{0}$•2,4,6... 0	0• $\underline{1}$• 1	4• $\underline{-3}$• 1
$op2_1$:41	0•41+1	1•41-0	1•41+2
x_1:41	0•41-0	1•41-1	1•41-0
op1 spacing	2•1,3,5... $\underline{-1}$•2,4,6... 1		
z	0	-1	-1

Chart 20. Family ${}^{1:2:2}_{w}S$.

	$^{1:2:3}{}_1S$	$^{1:2:3}{}_2S$	$^{1:2:3}{}_3S$
x_1	1	41	44
operands	43•1	43•41	43•47
x spacing	-1•1,3,5... <u>1</u>•2,4,6... 0	-1• <u>2</u>• 1	-3• <u>4</u>• 1
op2 spacing	0•1,3,5... <u>0</u>•2,4,6... 0	0• <u>1</u>• 1	-4• <u>5</u>• 1
$op2_1$:41	0•41+1	1•41-0	1•41+6
x_1:41	0•41+1	1•41-0	1•41+3
op1 spacing	-2•1,3,5... <u>3</u>•2,4,6... 1		
z	0	1	1

Chart 21. Family $^{1:2:3}{}_wS$.

	$^{1:2:4}{}_1S$	$^{1:2:4}{}_2S$	$^{1:2:4}{}_3S$
x_1	-2	-42	-45
operands	-43•-1	-43•-41	-43•-47
x spacing	-1•1,3,5... <u>1</u>•2,4,6... 0	-3• <u>2</u>• -1	-5• <u>4</u>• -1
op2 spacing	0•1,3,5... <u>0</u>•2,4,6... 0	-2• <u>1</u>• -1	-6• <u>5</u>• -1
$op2_1$:41	0•41-1	-1•41-0	-1•41-6
x_1:41	0•41-2	-1•41-1	-1•41-4
op1 spacing	-4•1,3,5... <u>3</u>•2,4,6... -1		
z	0	-1	-1

Chart 22. Family $^{1:2:4}{}_wS$.

	$1{:}2{:}5\ {}_1S$	$1{:}2{:}5\ {}_2S$	$1{:}2{:}5\ {}_3S$
x_1	2	44	49
operands	47•1	47•43	47•53
x spacing	-1•1,3,5... 1•2,4,6... 0	-3• 4• 1	-5• 6• 1
op2 spacing	0•1,3,5... 0•2,4,6... 0	-2• 3• 1	-6• 7• 1
$op2_1{:}41$	0•41+1	1•41+2	1•41+12
$x_1{:}41$	0•41+2	1•41+3	1•41+8
op1 spacing	-4•1,3,5... 5•2,4,6... 1		
z	0	1	1

Chart 23. Family $^{1:2:5}S$.

	$1{:}2{:}6\ {}_1S$	$1{:}2{:}6\ {}_2S$	$1{:}2{:}6\ {}_3S$
x_1	-3	-45	-50
operands	-47•-1	-47•-43	-47•-53
x spacing	-1•1,3,5... 1•2,4,6... 0	-5• 4• -1	-7• 6• -1
op2 spacing	0•1,3,5... 0•2,4,6... 0	-4• 3• -1	-8• 7• -1
$op2_1{:}41$	0•41-1	-1•41-2	-1•41-12
$x_1{:}41$	0•41-3	-1•41-4	-1•41-9
op1 spacing	-6•1,3,5... 5•2,4,6... -1		
z	0	-1	-1

Chart 24. Family $^{1:2:6}{}_wS$.

	${}^{1:3:1}_{\ \ \ 1}S$	${}^{1:3:1}_{\ \ \ 2}S$	${}^{1:3:1}_{\ \ \ 3}S$
x_1	81	-41	122
operands	41•163	41•41	41•367
x spacing	1•1,3,5... 1•2,4,6... 2	-1• 0• -1	1• 2• 3
op2 spacing	4•1,3,5... 0•2,4,6... 4	2• -1• 1	6• 3• 9
$op2_1$:41	4•41-1	1•41-0	9•41-2
x_1:41	2•41-1	-1•41-0	3•41-1
op1 spacing	0•1,3,5... 1•2,4,6... 1		
z	2	-1	3

Chart 25. Family ${}^{1:3:1}_{\ \ \ w}S$.

	${}^{1:3:2}_{\ \ \ 1}S$	${}^{1:3:2}_{\ \ \ 2}S$	${}^{1:3:2}_{\ \ \ 3}S$
x_1	81	-41	122
operands	41•163	41•41	41•367
x spacing	3•1,3,5... -1•2,4,6... 2	-1• 0• -1	5• -2• 3
op2 spacing	4•1,3,5... 0•2,4,6... 4	0• 1• 1	12• -3• 9
$op2_1$:41	4•41-1	1•41-0	9•41-2
x_1:41	2•41-1	-1•41-0	3•41-1
op1 spacing	2•1,3,5... -1•2,4,6... 1		
z	2	-1	3

Chart 26. Family ${}^{1:3:2}_{\ \ \ w}S$.

	$^{1:3:3}{}_1S$	$^{1:3:3}{}_2S$	$^{1:3:3}{}_3S$
x_1	244	122	611
operands	367•163	367•41	367•1019
x spacing	5•1,3,5... 1•2,4,6... 6	1• 2• 3	11• 4• 15
op2 spacing	4•1,3,5... 0•2,4,6... 4	0• 1• 1	20• 5• 25
$op2_1$:41	4•41-1	1•41-0	25•41-6
x_1:41	6•41-2	3•41-1	15•41-4
op1 spacing	6•1,3,5... 3•2,4,6... 9		
z	2	1	5

Chart 27. Family $^{1:3:3}{}_wS$.

	$^{1:3:4}{}_1S$	$^{1:3:4}{}_2S$	$^{1:3:4}{}_3S$
x_1	244	122	611
operands	367•163	367•41	367•1019
x spacing	7•1,3,5... -1•2,4,6... 6	5• -2• 3	19• -4• 15
op2 spacing	4•1,3,5... 0•2,4,6... 4	2• -1• 1	30• -5• 25
$op2_1$:41	4•41-1	1•41-0	25•41-6
x_1:41	6•41-2	3•41-1	15•41-4
op1 spacing	12•1,3,5... -3•2,4,6... 9		
z	2	1	5

Chart 28. Family $^{1:3:4}{}_wS$.

	$^{1:3:5}{}_{1}S$	$^{1:3:5}{}_{2}S$	$^{1:3:5}{}_{3}S$
x_1	407	611	1426
operands	1019•163	1019•367	1019•1997
x spacing	9•1,3,5... 1•2,4,6... 10	11• 4• 15	29• 6• 35
op2 spacing	4•1,3,5... 0•2,4,6... 4	6• 3• 9	42• 7• 49
$op2_1$:41	4•41-1	9•41-2	49•4-12
x_1:41	10•41-3	15•41-4	35•41-9
op1 spacing	20•1,3,5... 5•2,4,6... 25		
z	2	3	7

Chart 29. Family $^{1:3:5}{}_{w}S$.

	$^{1:3:6}{}_{1}S$	$^{1:3:6}{}_{2}S$	$^{1:3:6}{}_{3}S$
x_1	407	611	1426
operands	1019•163	1019•367	1019•1997
x spacing	11•1,3,5... -1•2,4,6... 10	19• -4• 15	41• -6• 35
op2 spacing	4•1,3,5... 0•2,4,6... 4	12• -3• 9	56• -7• 49
$op2_1$:41	4•41-1	9•41-2	49•41-12
x_1:41	10•41-3	15•41-4	35•41-9
op1 spacing	30•1,3,5... -5•2,4,6... 25		
z	2	3	7

Chart 30. Family $^{1:3:6}{}_{w}S$.

	$^{1:4:1}{}_{1}S$	$^{1:4:1}{}_{2}S$	$^{1:4:1}{}_{3}S$
x_1	41	-1	82
operands	41•43	41•1	41•167
x spacing	-1•1,3,5... 2•2,4,6... 1	1• -1• 0	-1• 3• 2
op2 spacing	-2•1,3,5... 3•2,4,6... 1	0• 0• 0	-4• 8• 4
$op2_1$:41	1•41+2	0•41+1	4•41+3
x_1:41	1•41-0	0•41-1	2•41-0
op1 spacing	0•1,3,5... 1•2,4,6... 1		
z	1	0	2

Chart 31. Family $^{1:4:1}{}_{w}S$.

	$^{1:4:2}{}_{1}S$	$^{1:4:2}{}_{2}S$	$^{1:4:2}{}_{3}S$
x_1	1	-1	2
operands	1•43	-1•-41	1•47
x spacing	-1•1,3,5... 1•2,4,6... 0	1• -1• 0	-1• 1• 0
op2 spacing	-2•1,3,5... 3•2,4,6... 1	0• 1• 1	-4• 5• 1
$op2_1$:41	1•41+2	-1•41-0	1•41+6
x_1:41	0•41+1	0•41-1	0•41+2
op1 spacing	0•1,3,5... 0•2,4,6... 0		
z	1	-1	1

Chart 32. Family $^{1:4:2}{}_{w}S$.

	$^{1:4:3}_{1}S$	$^{1:4:3}_{2}S$	$^{1:4:3}_{3}S$
x_1	84	82	251
operands	167•43	167•41	167•379
x spacing	-3•1,3,5... $\frac{5}{2}$•2,4,6...	-1• $\frac{3}{2}$•	-7• $\frac{13}{6}$•
op2 spacing	-2•1,3,5... $\frac{3}{1}$•2,4,6...	0• $\frac{1}{1}$•	-12• $\frac{21}{9}$•
$op2_1$:41	1•41+2	1•41-0	9•41+10
x_1:41	2•41+2	2•41-0	6•41+5
op1 spacing	-4•1,3,5... $\frac{8}{4}$•2,4,6...		
z	1	1	3

Chart 33. Family $^{1:4:3}_{w}S$.

	$^{1:4:4}_{1}S$	$^{1:4:4}_{2}S$	$^{1:4:4}_{3}S$
x_1	44	2	91
operands	47•43	47•1	47•179
x spacing	-3•1,3,5... $\frac{4}{1}$•2,4,6...	-1• $\frac{1}{0}$•	-7• $\frac{9}{2}$•
op2 spacing	-2•1,3,5... $\frac{3}{1}$•2,4,6...	0• $\frac{0}{0}$•	-12• $\frac{16}{4}$•
$op2_1$:41	1•41+2	0•41+1	4•41+15
x_1:41	1•41+3	0•41+2	2•41+9
op1 spacing	-4•1,3,5... $\frac{5}{1}$•2,4,6...		
z	1	0	2

Chart 34. Family $^{1:4:4}_{w}S$.

	$^{1:4:5}{}_{1}S$	$^{1:4:5}{}_{2}S$	$^{1:4:5}{}_{3}S$
x_1	127	251	506
operands	379•43	379•167	379•677
x spacing	-5•1,3,5... $\frac{8}{3}$•2,4,6...	-7• $\frac{13}{6}$•	-17• $\frac{29}{12}$•
op2 spacing	-2•1,3,5... $\frac{3}{1}$•2,4,6...	-4• $\frac{8}{4}$•	-24• $\frac{40}{16}$•
$op2_1$:41	1•41+2	4•41+3	16•41+21
x_1:41	3•41+4	6•41+5	12•41+14
op1 spacing	-12•1,3,5... $\frac{21}{9}$•2,4,6...		
z	1	2	4

Chart 35. Family $^{1:4:5}{}_{w}S$.

	$^{1:4:6}{}_{1}S$	$^{1:4:6}{}_{2}S$	$^{1:4:6}{}_{3}S$
x_1	87	91	266
operands	179•43	179•47	179•397
x spacing	-5•1,3,5... $\frac{7}{2}$•2,4,6...	-7• $\frac{9}{2}$•	-17• $\frac{23}{6}$•
op2 spacing	-2•1,3,5... $\frac{3}{1}$•2,4,6...	-4• $\frac{5}{1}$•	-24• $\frac{33}{9}$•
$op2_1$:41	1•41+2	1•41+6	9•41+28
x_1:41	2•41+5	2•41+9	6•41+20
op1 spacing	-12•1,3,5... $\frac{16}{4}$•2,4,6...		
z	1	1	3

Chart 36. Family $^{1:4:6}{}_{w}S$.

	$^{1:5:1}{}_{1}S$	$^{1:5:1}{}_{2}S$	$^{1:5:1}{}_{3}S$
x_1	122	-82	163
operands	41•367	41•163	41•653
x spacing	1•1,3,5... 2•2,4,6... 3	-1• -1• -2	1• 3• 4
op2 spacing	6•1,3,5... 3•2,4,6... 9	4• 0• 4	8• 8• 16
$op2_1$:41	9•41-2	4•41-1	16•41-3
x_1:41	3•41-1	-2•41-0	4•41-1
op1 spacing	0•1,3,5... 1•2,4,6... 1		
z	3	-2	4

Chart 37. Family $^{1:5:1}{}_{w}S$.

	$^{1:5:2}{}_{1}S$	$^{1:5:2}{}_{2}S$	$^{1:5:2}{}_{3}S$
x_1	244	-82	407
operands	163•367	163•41	163•1019
x spacing	5•1,3,5... 1•2,4,6... 6	-1• -1• -2	9• 1• 10
op2 spacing	6•1,3,5... 3•2,4,6... 9	0• 1• 1	20• 5• 25
$op2_1$:41	9•41-2	1•41-0	25•41-6
x_1:41	6•41-2	-2•41-0	10•41-3
op1 spacing	4•1,3,5... 0•2,4,6... 4		
z	3	-1	5

Chart 38. Family $^{1:5:2}{}_{w}S$.

	${}^{1:5:3}_{1}S$	${}^{1:5:3}_{2}S$	${}^{1:5:3}_{3}S$
x_1	489	163	1142
operands	653•367	653•41	653•1999
x spacing	7•1,3,5... 5•2,4,6... 12	1• 3• 4	15• 13• 28
op2 spacing	6•1,3,5... 3•2,4,6... 9	0• 1• 1	28• 21• 49
$op2_1$:41	9•41-2	1•41-0	49•41-10
x_1:41	12•41-3	4•41-1	28•41-6
op1 spacing	8•1,3,5... 8•2,4,6... 16		
z	3	1	7

Chart 39. Family ${}^{1:5:3}_{w}S$.

	${}^{1:5:4}_{1}S$	${}^{1:5:4}_{2}S$	${}^{1:5:4}_{3}S$
x_1	611	407	1630
operands	1019•367	1019•163	1019•2609
x spacing	11•1,3,5... 4•2,4,6... 15	9• 1• 10	31• 9• 40
op2 spacing	6•1,3,5... 3•2,4,6... 9	4• 0• 4	48• 16• 64
$op2_1$:41	9•41-2	4•41-1	64•41-15
x_1:41	15•41-4	10•41-3	40•41-10
op1 spacing	20•1,3,5... 5•2,4,6... 25		
z	3	2	8

Chart 40. Family ${}^{1:5:4}_{w}S$.

	$^{1:5:5}{}_{1}S$	$^{1:5:5}{}_{2}S$	$^{1:5:5}{}_{3}S$
x_1	856	1142	2855
operands	1999•367	1999•653	1999•4079
x spacing	13•1,3,5... 8•2,4,6... 21	15• 13• 28	41• 29• 70
op2 spacing	6•1,3,5... 3•2,4,6... 9	8• 8• 16	60• 40• 100
$op2_1$:41	9•41-2	16•41-3	100•41-21
x_1:41	21•41-5	28•41-6	70•41-15
op1 spacing	28•1,3,5... 21•2,4,6... 49		
z	3	4	10

Chart 41. Family $^{1:5:5}{}_{w}S$.

	$^{1:5:6}{}_{1}S$	$^{1:5:6}{}_{2}S$	$^{1:5:6}{}_{3}S$
x_1	978	1630	3587
operands	2609•367	2609•1019	2609•4933
x spacing	17•1,3,5... 7•2,4,6... 24	31• 9• 40	65• 23• 88
op2 spacing	6•1,3,5... 3•2,4,6... 9	20• 5• 25	88• 33• 121
$op2_1$:41	9•41-2	25•41-6	121•41-28
x_1:41	24•41-6	40•41-10	88•41-21
op1 spacing	48•1,3,5... 16•2,4,6... 64		
z	3	5	11

Chart 42. Family $^{1:5:6}{}_{w}S$.

	$^{1:6:1}{}_{1}S$	$^{1:6:1}{}_{2}S$	$^{1:6:1}{}_{3}S$
x_1	82	-42	123
operands	41•167	41•43	41•373
x spacing	-1•1,3,5... 3•2,4,6... 2	1• -2• -1	-1• 4• 3
op2 spacing	-4•1,3,5... 8•2,4,6... 4	-2• 3• 1	-6• 15• 9
$op2_1$:41	4•41+3	1•41+2	9•41+4
x_1:41	2•41-0	-1•41-1	3•41-0
op1 spacing	0•1,3,5... 1•2,4,6... 1		
z	2	-1	3

Chart 43. Family $^{1:6:1}{}_{w}S$.

	$^{1:6:2}{}_{1}S$	$^{1:6:2}{}_{2}S$	$^{1:6:2}{}_{3}S$
x_1	84	-42	127
operands	43•167	43•41	43•379
x spacing	-3•1,3,5... 5•2,4,6... 2	1• -2• -1	-5• 8• 3
op2 spacing	-4•1,3,5... 8•2,4,6... 4	0• 1• 1	-12• 21• 9
$op2_1$:41	4•41+3	1•41-0	9•41+10
x_1:41	2•41+2	-1•41-1	3•41+4
op1 spacing	-2•1,3,5... 3•2,4,6... 1		
z	2	-1	3

Chart 44. Family $^{1:6:2}{}_{w}S$.

	${}^{1:6:3}_{1}S$	${}^{1:6:3}_{2}S$	${}^{1:6:3}_{3}S$
x_1	249	123	622
operands	373•167	373•41	373•1039
x spacing	-5•1,3,5... $\frac{11}{6}$•2,4,6...	-1• $\frac{4}{3}$•	-11• $\frac{26}{15}$•
op2 spacing	-4•1,3,5... $\frac{8}{4}$•2,4,6...	0• $\frac{1}{1}$•	-20• $\frac{45}{25}$•
$op2_1$:41	4•41+3	1•41-0	25•41+14
x_1:41	6•41+3	3•41-0	15•41+7
op1 spacing	-6•1,3,5... $\frac{15}{9}$•2,4,6...		
z	2	1	5

Chart 45. Family ${}^{1:6:3}_{w}S$.

	${}^{1:6:4}_{1}S$	${}^{1:6:4}_{2}S$	${}^{1:6:4}_{3}S$
x_1	251	127	630
operands	379•167	379•43	379•1049
x spacing	-7•1,3,5... $\frac{13}{6}$•2,4,6...	-5• $\frac{8}{3}$•	-19• $\frac{34}{15}$•
op2 spacing	-4•1,3,5... $\frac{8}{4}$•2,4,6...	-2• $\frac{3}{1}$•	-30• $\frac{55}{25}$•
$op2_1$:41	4•41+3	1•41+2	25•41+24
x_1:41	6•41+5	3•41+4	15•41+15
op1 spacing	-12•1,3,5... $\frac{21}{9}$•2,4,6...		
z	2	1	5

Chart 46. Family ${}^{1:6:4}_{w}S$.

	$^{1:6:5}{}_{1}S$	$^{1:6:5}{}_{2}S$	$^{1:6:5}{}_{3}S$
x_1	416	622	1455
operands	1039•167	1039•373	1039•2039
x spacing	-9•1,3,5... 19•2,4,6... 10	-11• 26• 15	-29• 64• 35
op2 spacing	-4•1,3,5... 8•2,4,6... 4	-6• 15• 9	-42• 91• 49
$op2_1$:41	4•41+3	9•41+4	49•41+30
x_1:41	10•41+6	15•41+7	35•41+20
op1 spacing	-20•1,3,5... 45•2,4,6... 25		
z	2	3	7

Chart 47. Family $^{1:6:5}{}_{w}S$.

	$^{1:6:6}{}_{1}S$	$^{1:6:6}{}_{2}S$	$^{1:6:6}{}_{3}S$
x_1	418	630	1467
operands	1049•167	1049•379	1049•2053
x spacing	-11•1,3,5... 21•2,4,6... 10	-19• 34• 15	-41• 76• 35
op2 spacing	-4•1,3,5... 8•2,4,6... 4	-12• 21• 9	-56• 105• 49
$op2_1$:41	4•41+3	9•41+10	49•41+44
x_1:41	10•41+8	15•41+15	35•41+32
op1 spacing	-30•1,3,5... 55•2,4,6... 25		
z	2	3	7

Chart 48. Family $^{1:6:6}{}_{w}S$.

	$^{1:7:1}{}_{1}S$	$^{1:7:1}{}_{2}S$	$^{1:7:1}{}_{3}S$
x_1	163	-123	204
operands	41•653	41•367	41•1021
x spacing	1•1,3,5... 3•2,4,6... 4	-1• -2• -3	1• 4• 5
op2 spacing	8•1,3,5... 8•2,4,6... 16	6• 3• 9	10• 15• 25
$op2_1$:41	16•41-3	9•41-2	25•41-4
x_1:41	4•41-1	-3•41-0	5•41-1
op1 spacing	0•1,3,5... 1•2,4,6... 1		
z	4	-3	5

Chart 49. Family $^{1:7:1}{}_{w}S$.

	$^{1:7:2}{}_{1}S$	$^{1:7:2}{}_{2}S$	$^{1:7:2}{}_{3}S$
x_1	489	-123	856
operands	367•653	367•41	367•1999
x spacing	7•1,3,5... 5•2,4,6... 12	-1• -2• -3	13• 8• 21
op2 spacing	8•1,3,5... 8•2,4,6... 16	0• 1• 1	28• 21• 49
$op2_1$:41	16•41-3	1•41-0	49•41-10
x_1:41	12•41-3	-3•41-0	21•41-5
op1 spacing	6•1,3,5... 3•2,4,6... 9		
z	4	-1	7

Chart 50. Family $^{1:7:2}{}_{w}S$.

	$1{:}7{:}3\;{}_1\underline{S}$	$1{:}7{:}3\;{}_2\underline{S}$	$1{:}7{:}3\;{}_3\underline{S}$
x_1	816	204	1837
operands	1021•653	1021•41	1021•3307
x spacing	9•1,3,5... 11•2,4,6... 20	1• 4• 5	19• 26• 45
op2 spacing	8•1,3,5... 8•2,4,6... 16	0• 1• 1	36• 45• 81
$op2_1{:}41$	16•41-3	1•41-0	81•41-14
$x_1{:}41$	20•41-4	5•41-1	45•41-8
op1 spacing	10•1,3,5... 15•2,4,6... 25		
z	4	1	9

Chart 51. Family $1{:}7{:}3\;{}_wS$.

	$1{:}7{:}4\;{}_1\underline{S}$	$1{:}7{:}4\;{}_2\underline{S}$	$1{:}7{:}4\;{}_3\underline{S}$
x_1	1142	856	3141
operands	1999•653	1999•367	1999•4937
x spacing	15•1,3,5... 13•2,4,6... 28	13• 8• 21	43• 34• 77
op2 spacing	8•1,3,5... 8•2,4,6... 16	6• 3• 9	66• 55• 121
$op2_1{:}41$	16•41-3	9•41-2	121•41-24
$x_1{:}41$	28•41-6	21•41-5	77•41-16
op1 spacing	28•1,3,5... 21•2,4,6... 49		
z	4	3	11

Chart 52. Family $1{:}7{:}4\;{}_wS$.

	$^{1:7:5}{}_{1}S$	$^{1:7:5}{}_{2}S$	$^{1:7:5}{}_{3}S$
x_1	1469	1837	4776
operands	3307•653	3307•1021	3307•6899
x spacing	17•1,3,5... 19•2,4,6... 36	19• 26• 45	53• 64• 117
op2 spacing	8•1,3,5... 8•2,4,6... 16	10• 15• 25	78• 91• 169
$op2_1$:41	16•41-3	25•41-4	169•41-30
x_1:41	36•41-7	45•41-8	117•41-21
op1 spacing	36•1,3,5... 45•2,4,6... 81		
z	4	5	13

Chart 53. Family $^{1:7:5}{}_{w}S$.

	$^{1:7:6}{}_{1}S$	$^{1:7:6}{}_{2}S$	$^{1:7:6}{}_{3}S$
x_1	1795	3141	6732
operands	4937•653	4937•1999	4937•9181
x spacing	23•1,3,5... 21•2,4,6... 44	43• 34• 77	89• 76• 165
op2 spacing	8•1,3,5... 8•2,4,6... 16	28• 21• 49	120• 105• 225
$op2_1$:41	16•41-3	49•41-10	225•41-44
x_1:41	44•41-9	77•41-16	165•41-33
op1 spacing	66•1,3,5... 55•2,4,6... 121		
z	4	7	15

Chart 54. Family $^{1:7:6}{}_{w}S$.

	$^{1:8:1}{}_{1}S$	$^{1:8:1}{}_{2}S$	$^{1:8:1}{}_{3}S$
x_1	123	-83	164
operands	41•373	41•167	41•661
x spacing	-1•1,3,5... 4•2,4,6... 3	1• -3• -2	-1• 5• 4
op2 spacing	-6•1,3,5... 15•2,4,6... 9	-4• 8• 4	-8• 24• 16
$op2_1$:41	9•41+4	4•41+3	16•41+5
x_1:41	3•41-0	-2•41-1	4•41-0
op1 spacing	0•1,3,5... 1•2,4,6... 1		
z	3	-2	4

Chart 55. Family $^{1:8:1}{}_{w}S$.

	$^{1:8:2}{}_{1}S$	$^{1:8:2}{}_{2}S$	$^{1:8:2}{}_{3}S$
x_1	249	-83	416
operands	167•373	167•41	167•1039
x spacing	-5•1,3,5... 11•2,4,6... 6	1• -3• -2	-9• 19• 10
op2 spacing	-6•1,3,5... 15•2,4,6... 9	0• 1• 1	-20• 45• 25
$op2_1$:41	9•41+4	1•41-0	25•41+14
x_1:41	6•41+3	-2•41-1	10•41+6
op1 spacing	-4•1,3,5... 8•2,4,6... 4		
z	3	-1	5

Chart 56. Family $^{1:8:2}{}_{w}S$.

	$\underline{1:8:3}_{1}S$	$\underline{1:8:3}_{2}S$	$\underline{1:8:3}_{3}S$
x_1	496	164	1157
operands	661•373	661•41	661•2027
x spacing	-7•1,3,5... 19•2,4,6... 12	-1• 5• 4	-15• 43• 28
op2 spacing	-6•1,3,5... 15•2,4,6... 9	0• 1• 1	-28• 77• 49
$op2_1$:41	9•41+4	1•41-0	49•41+18
x_1:41	12•41+4	4•41-0	28•41+9
op1 spacing	-8•1,3,5... 24•2,4,6... 16		
z	3	1	7

Chart 57. Family $1{:}8{:}3\ _{w}S$

	$\underline{1:8:4}_{1}S$	$\underline{1:8:4}_{2}S$	$\underline{1:8:4}_{3}S$
x_1	622	416	1661
operands	1039•373	1039•167	1039•2657
x spacing	-11•1,3,5... 26•2,4,6... 15	-9• 19• 10	-31• 71• 40
op2 spacing	-6•1,3,5... 15•2,4,6... 9	-4• 8• 4	-48• 112• 64
$op2_1$:41	9•41+4	4•41+3	64•41+33
x_1:41	15•41+7	10•41+6	40•41+21
op1 spacing	-20•1,3,5... 45•2,4,6... 25		
z	3	2	8

Chart 58. Family $1{:}8{:}4\ _{w}S$.

	$^{1:8:5}{}_1S$	$^{1:8:5}{}_2S$	$^{1:8:5}{}_3S$
x_1	869	1157	2896
operands	2027•373	2027•661	2027•4139
x spacing	-13•1,3,5... 34•2,4,6... 21	-15• 43• 28	-41• 111• 70
op2 spacing	-6•1,3,5... 15•2,4,6... 9	-8• 24• 16	-60• 160• 100
$op2_1$:41	9•41+4	16•41+5	100•41+39
x_1:41	21•41+8	28•41+9	70•41+26
op1 spacing	-28•1,3,5... 77•2,4,6... 49		
z	3	4	10

Chart 59. Family $^{1:85}{}_wS$.

	$^{1:8:6}{}_1S$	$^{1:8:6}{}_2S$	$^{1:8:6}{}_3S$
x_1	995	1661	3652
operands	2657•373	2657•1039	2657•5021
x spacing	-17•1,3,5... 41•2,4,6... 24	-31• 71• 40	-65• 153• 88
op2 spacing	-6•1,3,5... 15•2,4,6... 9	-20• 45• 25	-88• 209• 121
$op2_1$:41	9•41+4	25•41+14	121•41+60
x_1:41	24•41+11	40•41+21	88•41+44
op1 spacing	-48•1,3,5... 112•2,4,6... 64		
z	3	5	11

Chart 60. Family $^{1:8:6}{}_wS$

	${}^{1:9:1}_{1}S$	${}^{1:9:1}_{2}S$	${}^{1:9:1}_{3}S$
x_1	204	-164	245
operands	41•1021	41•653	41•1471
x spacing	1•1,3,5... 4•2,4,6... 5	-1• -3• -4	1• 5• 6
op2 spacing	10•1,3,5... 15•2,4,6... 25	8• 8• 16	12• 24• 36
$op2_1$:41	25•41-4	16•41-3	36•41-5
x_1:41	5•41-1	-4•41-0	6•41-1
op1 spacing	0•1,3,5... 1•2,4,6... 1		
z	5	-4	6

Chart 61. Family ${}^{1:9:1}_{w}S$.

	${}^{1:9:2}_{1}S$	${}^{1:9:2}_{2}S$	${}^{1:9:2}_{3}S$
x_1	816	-164	1469
operands	653•1021	653•41	653•3307
x spacing	9•1,3,5... 11•2,4,6... 20	-1• -3• -4	17• 19• 36
op2 spacing	10•1,3,5... 15•2,4,6... 25	0• 1• 1	36• 45• 81
$op2_1$:41	25•41-4	1•41-0	81•41-14
x_1:41	20•41-4	-4•41-0	36•41-7
op1 spacing	8•1,3,5... 8•2,4,6... 16		
z	5	-1	9

Chart 62. Family ${}^{1:9:2}_{w}S$.

	$^{1:9:3}{}_{1}S$	$^{1:9:3}{}_{2}S$	$^{1:9:3}{}_{3}S$
x_1	1225	245	2696
operands	1471•1021	1471•41	1471•4943
x spacing	11•1,3,5... 19•2,4,6... 30	1• 5• 6	23• 43• 66
op2 spacing	10•1,3,5... 15•2,4,6... 25	0• 1• 1	44• 77• 121
$op2_1$:41	25•41-4	1•41-0	121•41-18
x_1:41	30•41-5	6•41-1	66•41-10
op1 spacing	12•1,3,5... 24•2,4,6... 36		
z	5	1	11

Chart 63. Family $^{1:9:3}{}_{w}S$.

	$^{1:9:4}{}_{1}S$	$^{1:9:4}{}_{2}S$	$^{1:9:4}{}_{3}S$
x_1	1837	1469	5144
operands	3307•1021	3307•653	3307•8003
x spacing	19•1,3,5... 26•2,4,6... 45	17• 19• 36	55• 71• 126
op2 spacing	10•1,3,5... 15•2,4,6... 51	8• 8• 61	84• 112• 196
$op2_1$:41	25•41-4	16•41-3	196•41-33
x_1:41	45•41-8	36•41-7	126•41-22
op1 spacing	36•1,3,5... 45•2,4,6... 81		
z	5	4	14

Chart 64. Family $^{1:9:4}{}_{w}S$.

	$\frac{1:9:5}{1}S$	$\frac{1:9:5}{2}S$	$\frac{1:9:5}{3}S$
x_1	2246	2696	7189
operands	4943•1021	4943•1471	4943•10457
x spacing	21•1,3,5... 34•2,4,6... 55	23• 43• 66	65• 111• 176
op2 spacing	10•1,3,5... 15•2,4,6... 25	12• 24• 36	96• 160• 256
$op2_1$:41	25•41-4	36•41-5	256•41-39
x_1:41	55•41-9	66•41-10	176•41-27
op1 spacing	44•1,3,5... 77•2,4,6... 121		
z	5	6	16

Chart 65. Family $^{1:9:5}{}_{w}S$.

	$\frac{1:9:6}{1}S$	$\frac{1:9:6}{2}S$	$\frac{1:9:6}{3}S$
x_1	2858	5144	10861
operands	8003•1021	8003•3307	8003•14741
x spacing	29•1,3,5... 41•2,4,6... 70	55• 71• 126	113• 153• 266
op2 spacing	10•1,3,5... 15•2,4,6... 25	36• 45• 81	152• 209• 361
$op2_1$:41	25•41-4	81•41-14	361•41-60
x_1:41	70•41-12	126•41-22	266•41-45
op1 spacing	84•1,3,5... 112•2,4,6... 196		
z	5	9	19

Chart 66. Family $^{1:9:6}{}_{w}S$.

	$^{1:3:6}{}_{1}S$	$^{1:3:6}{}_{2}S$	$^{1:3:6}{}_{3}S$	$^{1:3:6}{}_{4}S$	$^{1:3:6}{}_{5}S$
x_1	407	611	1426	1630	2445
operands	1019•163	1019•367	1019•1997	1019•2609	1019•5869
x spacing	11•1,3,5... -1•2,4,6... 10	19• -4• 15	41• -6• 35	49• -9• 40	71• -11• 60
op2 spacing	4•1,3,5... 0•2,4,6... 4	12• -3• 9	56• -7• 49	80• -16• 64	168• -24• 144
$op2_1$:41	4•41-1	9•41-2	49•41-12	64•41-15	144•41-35
x_1:41	10•41-3	15•41-4	35•41-9	40•41-10	60•41-15
op1 spacing	30•1,3,5... -5•2,4,6... 25				
z	2	3	7	8	12

Chart 67. Family $^{1:3:6}{}_{w}S$.

	$^{1:3:6}{}_{1}S$		$^{1:3:6}{}_{2}S$		$^{1:3:6}{}_{3}S$		$^{1:3:6}{}_{4}S$		$^{1:3:6}{}_{5}S$
x_1 operands	407 1019•163		611 1019•367		1426 1019•1997		1630 1019•2609		2445 1019•5869
x spacing	11•1,3,5... -1•2,4,6... 10	8 -3	19• -4• 15	22 -2	41• -6• 35	8 -3	49• -9• 40	22 -2	71• -11• 60
op2 spacing	4•1,3,5... 0•2,4,6... 4	8•1 -3•1	12• -3• 9	22•2 -2•2	56• -7• 49	8•3 -3•3	80• -16• 64	22•4 -2•4	168• -24• 144
$op2_1$:41	4•41-1	1•1	9•41-2	5•2	49•41-12	1•3	64•41-15	5•4	144•41-35
x_1:41	10•41-3	1	15•41-4	5	35•41-9	1	40•41-10	5	60•41-15
op1 spacing	30•1,3,5... -5•2,4,6... 25								
z	2		3		7		8		12

Chart 68. Family $^{1:3:6}{}_{w}S$ with parameter value spacings across series.

	$^{1:2:3}{}_1S$	$^{1:2:3}{}_2S$	$^{1:2:3}{}_3S$	$^{1:2:3}{}_4S$
x_1	1	41	44	84
operands	43•1	43•41	43•47	43•167
x spacing	-1•1,3,5... 1•2,4,6... 0	-1• 2• 1	-3• 4• 1	-3• 5• 2
op2 spacing	0•1,3,5... 0•2,4,6... 0	0• 1• 1	-4• 5• 1	-4• 8• 4
$op2_1$:41	0•41+1	1•41-0	1•41+6	4•41+3
x_1:41	0•41+1	1•41-0	1•41+3	2•41+2
op1 spacing	-2•1,3,5... 3•2,4,6... 1			
z	0	1	1	2

Chart 69. Family $^{1:2:3}{}_wS$.

Appendix B -- Casio Calculator Programs

Program 1 - Given a factor pair, find the x that gives their product when it is plugged into $x^2 + x + 41$

```
Mcl:                         /* clear memory
"A":?→A:                     /* prompt for and store factor A
"B":?→B:                     /* prompt for and store factor B
(√(AxBx4-163)-1)÷2▲          /* find x and output it
```

Program 2 - Given the initial absolute x and the next op1 in a series, find that next factor pair's absolute x and op2

```
Lbl 0:                       /* label for location 0
Mcl:                         /* clear memory
"F":?→A:                     /* prompt for and store initial abs. x
"OP1":?→B:                   /* prompt for and store next op1
Lbl 1:                       /* label for location 1
(A²+A+41)÷B→C:               /* calc. x² + x + 41 and divide by op1
Frac C>0»Goto 2:             /* if non-0 remainder then go to loc 2
C▲A▲Goto 0:                  /* else output quotient (op2), x; next
Lbl 2:                       /* label for location 2
A+1→A:                       /* increment x
Goto 1:                      /* go to loc 1 to see if x now good
```

Program 3 - For any factor pair series w in family v in first-plane child plane u, the following parameters and values are found and displayed in succession for input u, v, and w:

$op1_1$, $c_{od}op1$, $c_{ev}op1$, z, $op2_1$, ±, $c_{od}op2$, and $c_{ev}op2$.

The user is then prompted for an x value (relative x within the series. For example the first factor pair in the series has relative x = 1). The program then displays $op1_x$ and $op2_x$ in succession. The x prompt then appears again.

(To see successive answers, the EXE key is pressed once per answer. To exit the program, after an answer appears, press AC.)

```
     Step
Mcl:                         /* clear memory
"U":?→U:                     /* prompt for and store u (plane)
"V":?→V:                     /* prompt for and store v (family)
"W":?→W:                     /* prompt for and store w (series)
```

```
U÷2→F:                          /* start calculating A_u
F-Frac F→A:                     /* finish calculating, store in A
A→B:                            /* copy A_u to B_u
Frac F=0»B-1→B:                 /* if u even, decrement B_u
A^2→C:                          /* calculate C_u
B^2+B→D:                        /* calculate D_u
C-2xA→E:                        /* calculate E_u
B^2-B→F:                        /* calculate F_u
2x(B-A)+1→G:                    /* calculate G_u
2x(D-C)+1→H:                    /* calculate H_u
2x(F-E)+1→I:                    /* calculate I_u
81x(G+A)+A→J:                   /* calculate J_u
40x(H+C)+D+1→K:                 /* calculate K_u
40x(I+E-2)+F→L:                 /* calculate L_u
V÷2→F:                          /* start calculating A_v
F-Frac F→O:                     /* finish calculating, store in O
O→P:                            /* copy A_v to B_v
Frac F=0»P-1→P:                 /* if v even, decrement B_v
O^2→Q:                          /* calculate C_v
P^2+P→R:                        /* calculate D_v
Q-2xO→S:                        /* calculate E_v
P^2-P→T:                        /* calculate F_v
2x(P-O)+1→X:                    /* calculate G_v
W÷2→F:                          /* start calculating A_w
F-Frac F→Y:                     /* finish calculating, store in Y
Y→Z:                            /* copy A_w to B_w
Frac F=0»Z-1→Z:                 /* if w even, decrement B_w
Y^2→B:                          /* calculate C_w (we're done with B_u
Z^2+Z→C:                        /* calculate D_w (we're done with C_u
2x(Z-Y)+1→D:                    /* calculate G_w (we're done with D_u
OxY+D→M:                        /* calculate M_vw
JxR+LxQ+41→H▲                   /* calculate and display op1_1
2xGxR+(I-1)xQ→I▲                /* calculate and display c_od op1
2xAxR+(E-1)xQ+1→U▲              /* calculate and display c_ev op1
(G+A)xM+XxY→N▲                  /* calculate and display z
(2xLxO+J(2xP+1))xC+(LxS+J(T-1)+41)xB+K→V▲   /* find and show op2_1
(3xG-A)÷2xM-(X÷2)xY→F▲          /* calculate and display ±
(NH÷2+F)xN→W▲                   /* calculate and display c_od op2
(N÷2-F)xN→Z▲                    /* calculate and display c_ev op2
Lbl 1:                          /* label for location 1
"X":?→X:                        /* prompt for and store relative x
X÷2→F:                          /* start calculating A_x
F-Frac F→A:                     /* finish calculating, store in A
A→B:                            /* copy A_x to B_x
Frac F=0»B-1→B:                 /* if x even, decrement B_x
A^2→C:                          /* calculate C_x
B^2+B→D:                        /* calculate D_x
H+CxI+DxU▲                      /* calculate and display op1_x
V+CxW+DxZ▲                      /* calculate and display op2_x
Goto 1:                         /* go to prompt for next x
```

Note: memory registers in the Casio are labeled "A" to "Z" -- rather than buy more registers at the expense of the number of program steps able to be stored, which is one option, we have elected to judiciously reuse some memory registers once they have served their purpose and their previous contents have been utilized. Several of the registers in this program have been used as many as three times (B, C, D, and F).

Program 4 -- Find the next op2 in a factor pair series, given the next op1, the current op2, and a direction of motion

```
/* A, B, and C are entered manually before execution is begun for
/* the first time.  After the first run, only A need be entered
/* to get subsequent results.

/* A = next op1        (key in: op1 value, →, alpha, A)
/* B = current op2     (key in: op2 value, →, alpha, B)
/* C = direction       (key in: 1 or -1, →, alpha, C)

Lbl 1:                              /* label for location 1
Frac ((√(AxBx4-163)-1)÷2)=0»B▲      /* if op1 • op2 → f(n), stop
B+C→B:                              /* else calc. next op2 candidate
Goto 1:                             /* try op1 • new op2 candidate
```