LINES OF PRIMES!

Michael J. Sussna

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Dr. Charles Giardina
Fairleigh Dickinson University
Computer Science Masters Program

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1. TRIANGULAR ARRAY PRIME LINES

A prime number is a whole number greater than 1 which is not a product of any whole numbers greater than 1. What patterns, if any, are recognizable when prime numbers are singled out in an array where the whole numbers are listed in a triangular way such as the following?

```
            1
         2     3
      6     5     4
   7     8     9     10
          etc.
```

Figure 1.

This was a question I asked myself after being inspired by the cover of the March, 1964 issue of Scientific American to try my hand at finding interesting prime number patterns via new ways of listing the whole numbers. There were indeed patterns generated, and though I turned 14 that March, it was not until my sophomore year at the University of Southern California five years later that I recognized them.

Before I discuss the patterns I eventually saw, I wish to briefly explain why there was a five year delay before I did see them.

As I was preparing my original triangular array of numbers, I made the near-disastrous mistake of imposing patterns on the primes already singled out, in the following manner: I connected any two primes either vertically or horizontally one space apart:

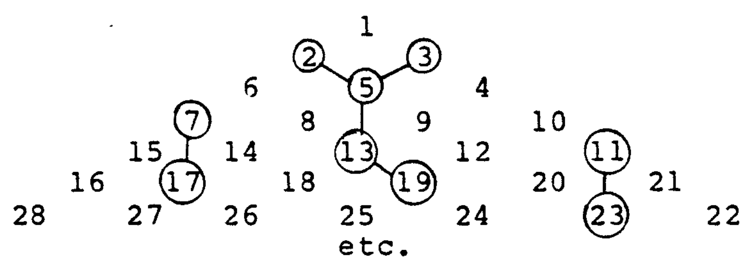

Figure 2.

I presumptuously called the random structures thus formed a "prime-stellar" relationship. What I meant was simply that each group of connected primes could be likened to star systems. (Most star systems have more than one sun, our solar system being an exception, having only one.) Actually, looking back, I guess I can't be blamed too severely for that approach (connecting neighboring primes) since the scheme detailed in the "Mathematical Games" section of that March, 1964 issue of Scientific American did the same type of thing with the primes when the whole numbers were listed in a spiral thus:

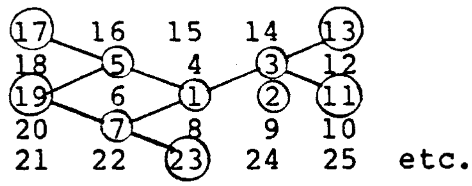

Figure 3.

At any rate, by the time I had placed the primes up to about the number 2,000 in my triangular array, the array was so cluttered with these extraneous "patterns" that I could no longer hope to see any real patterns there. (See figure 4.)

Indeed, it was not until I returned to the triangular array approach some five years later that I finally succeeded in merely completing the array before attempting pattern recognition.

This time my approach was richly rewarded. When the array of dots representing primes is turned at the proper angle and viewed on edge, lo and behold a series of parallel, straight, unbroken lines of primes leaps out at you! (See figures 5 and 6).

When the array is looked at similarly from the other side of the triangle, more lines appear. (See figure 7.)

When both sets of lines are taken together, all five such straight lines are evident. (See figure 8.)

I soon discovered a way to enhance this effect while at the same time consolidating the sets of patterns into a single set of parallel lines. This was accomplished by rearraying the prime dots according to a slightly different counting scheme:

```
            1
         2     3
      4     5     6
   7     8     9     10
          etc.
```

Figure 9.

(See figure 10.)

This form of the array eventually evolved into the cur-

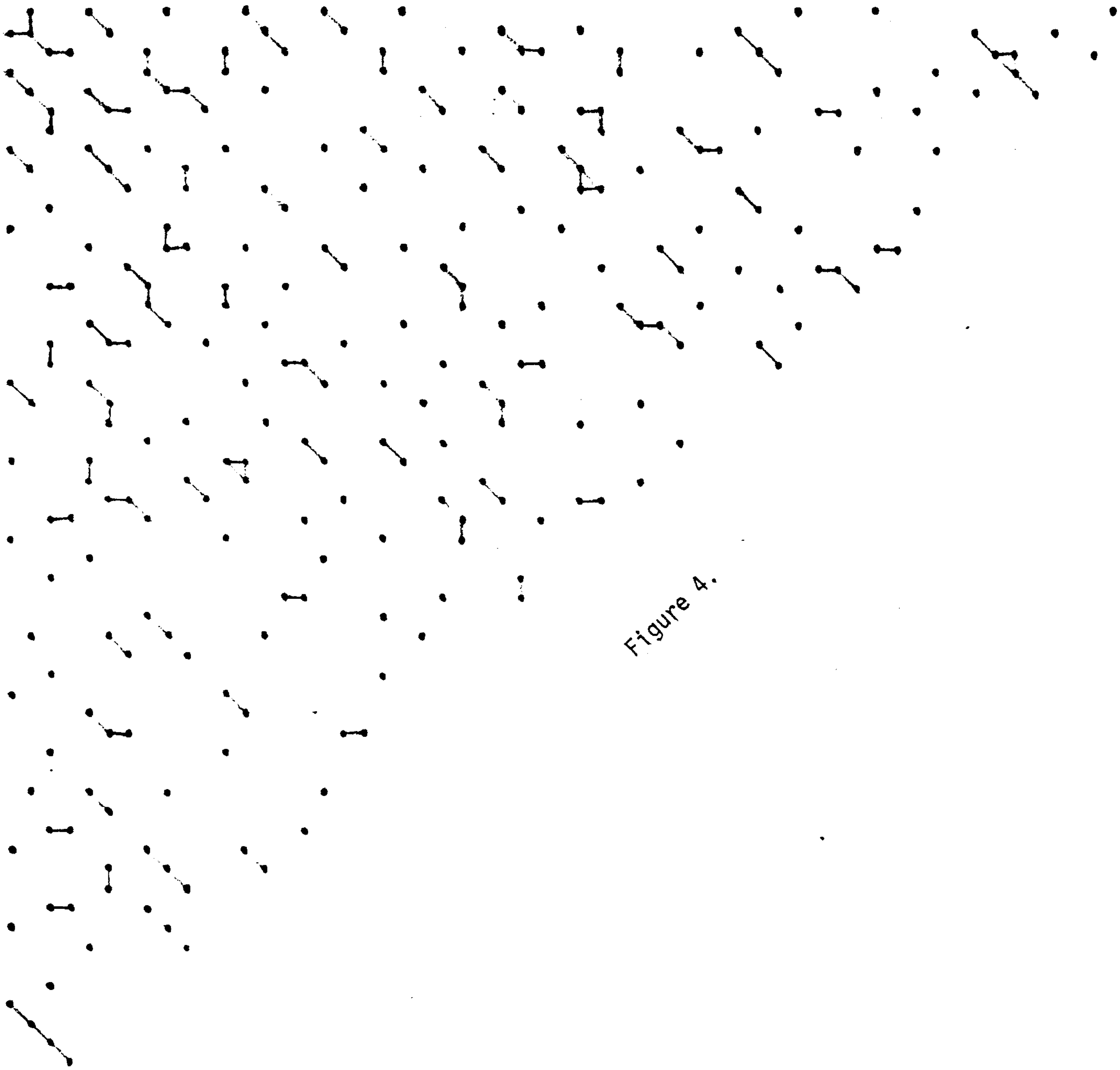

Figure 4.

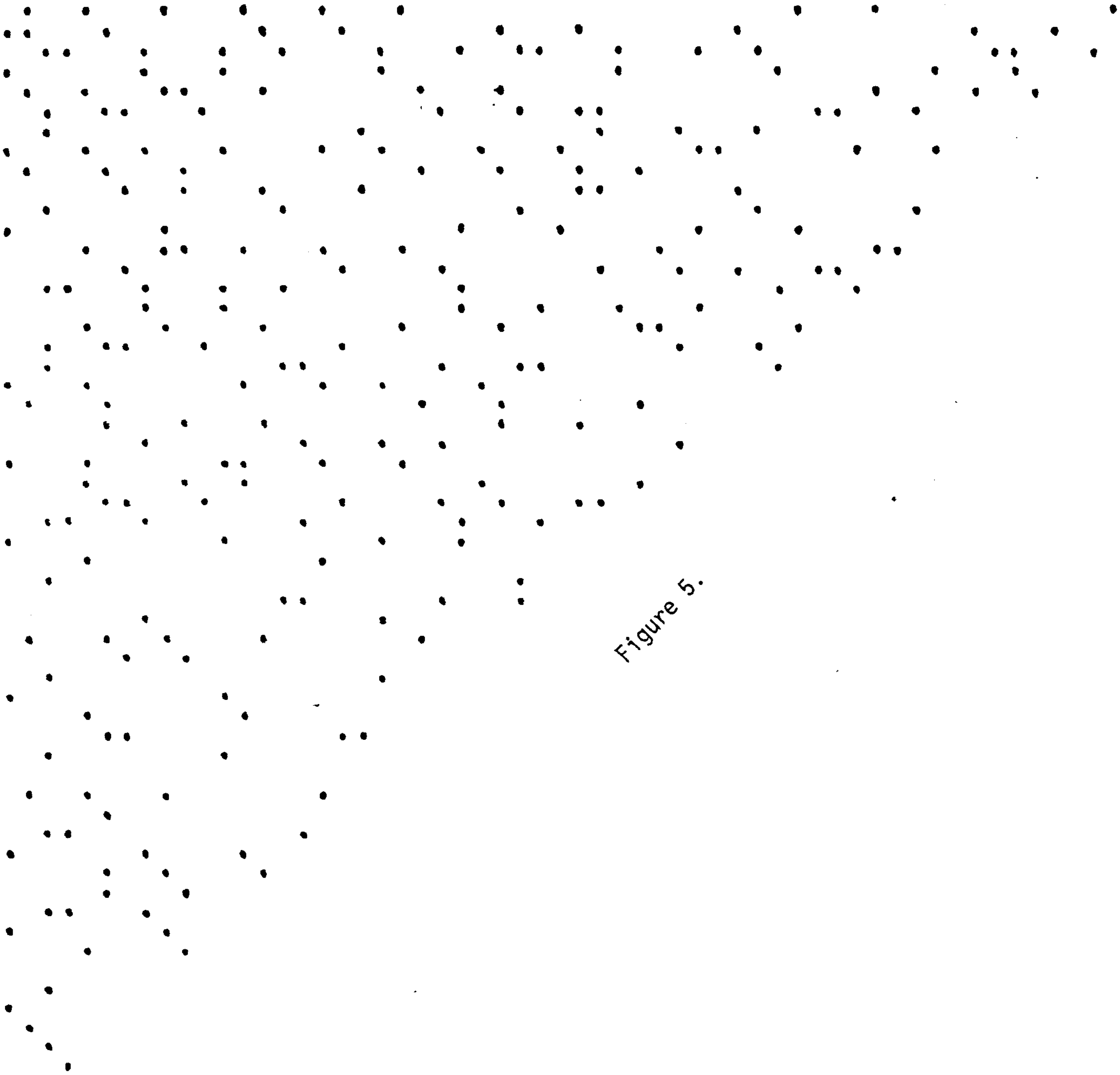

Figure 5.

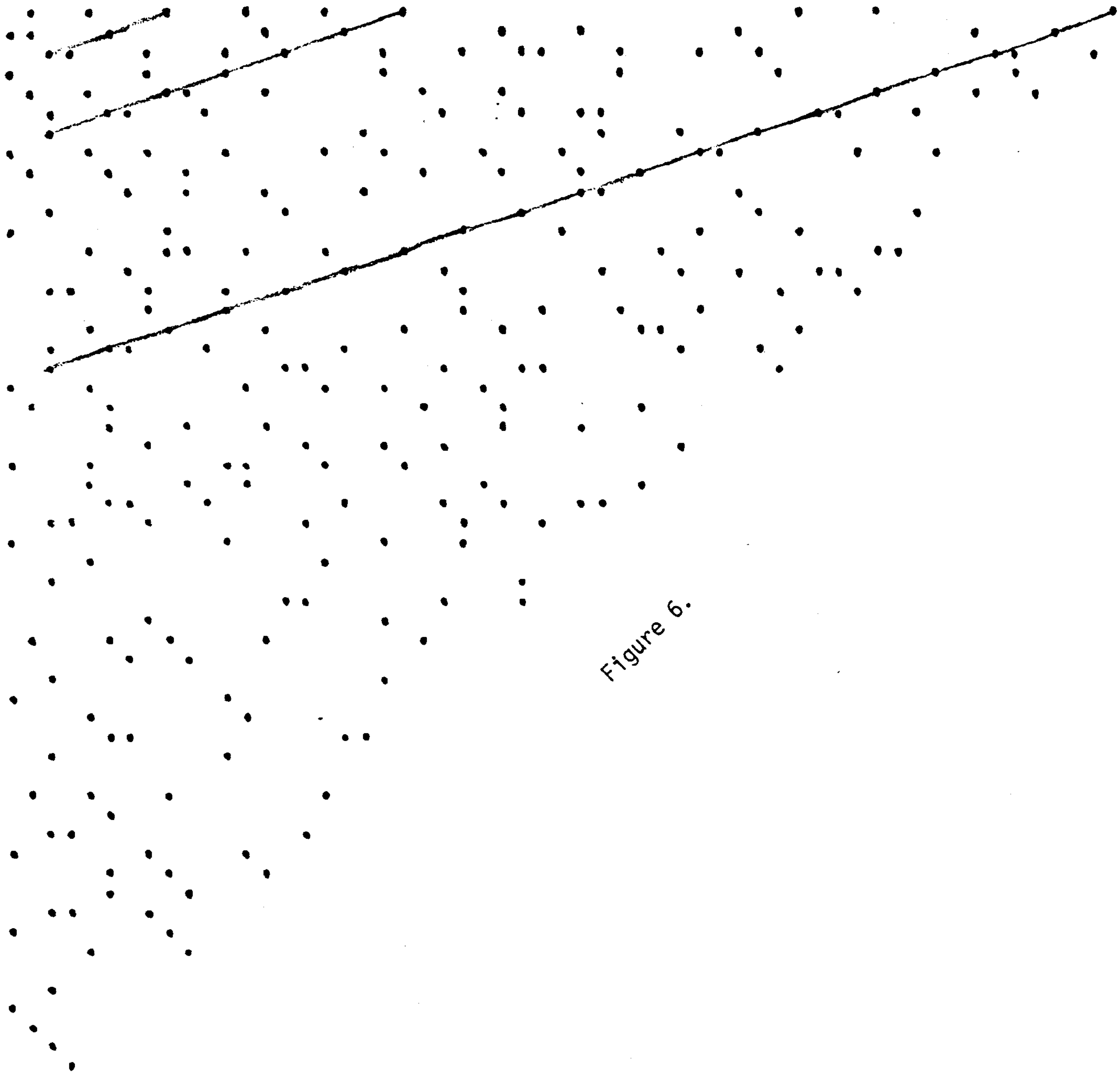

Figure 6.

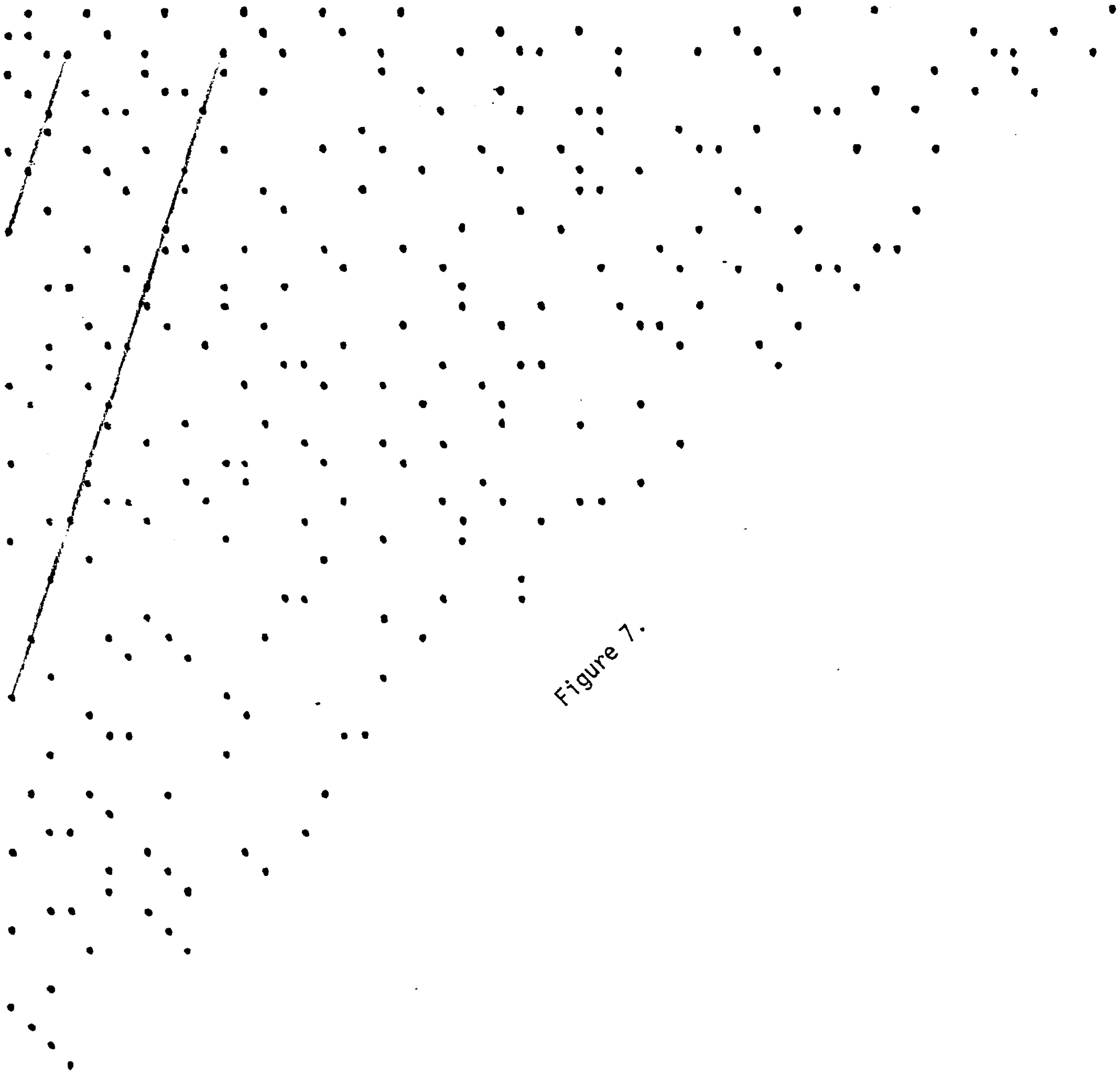

Figure 7.

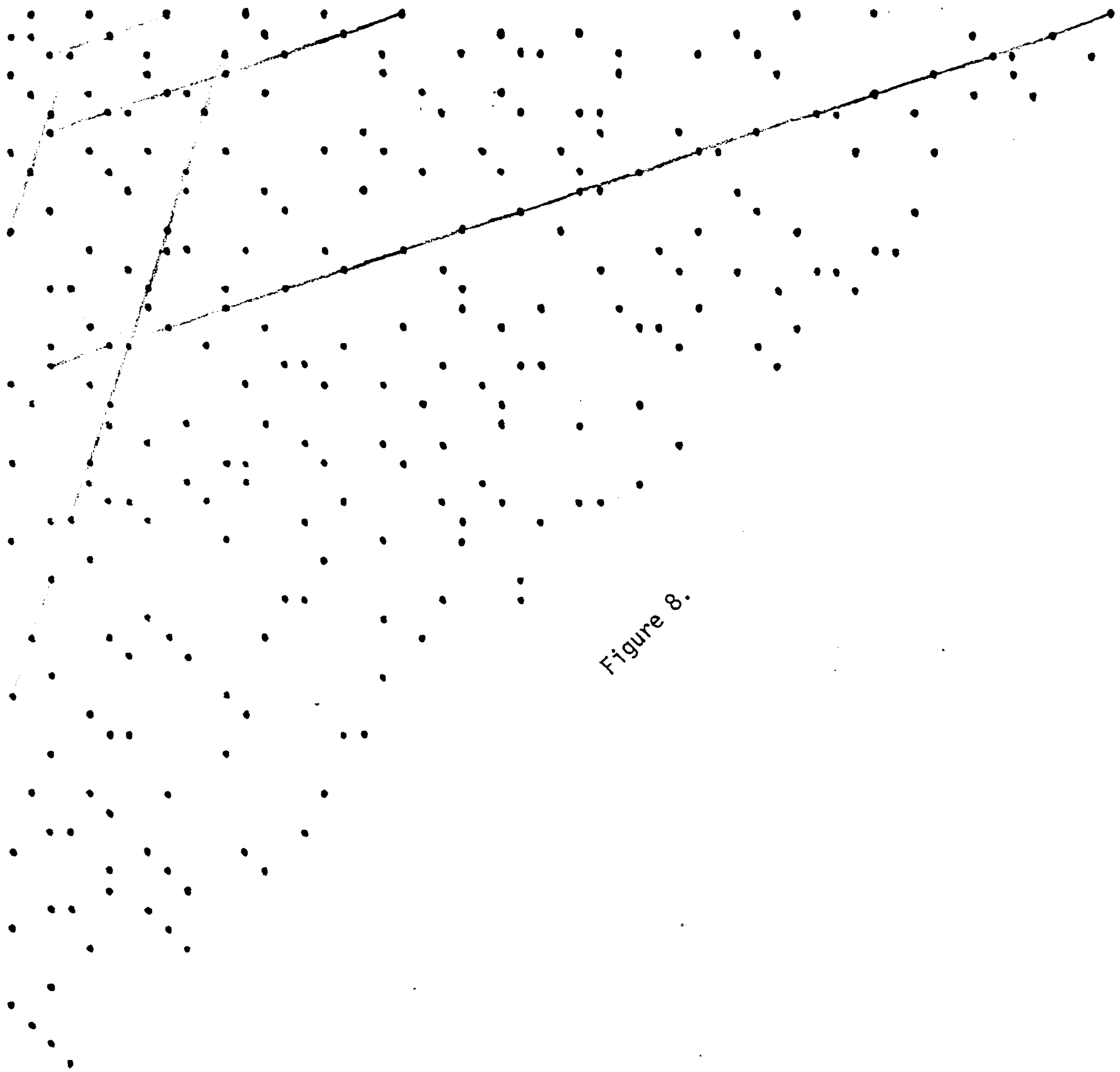

Figure 8.

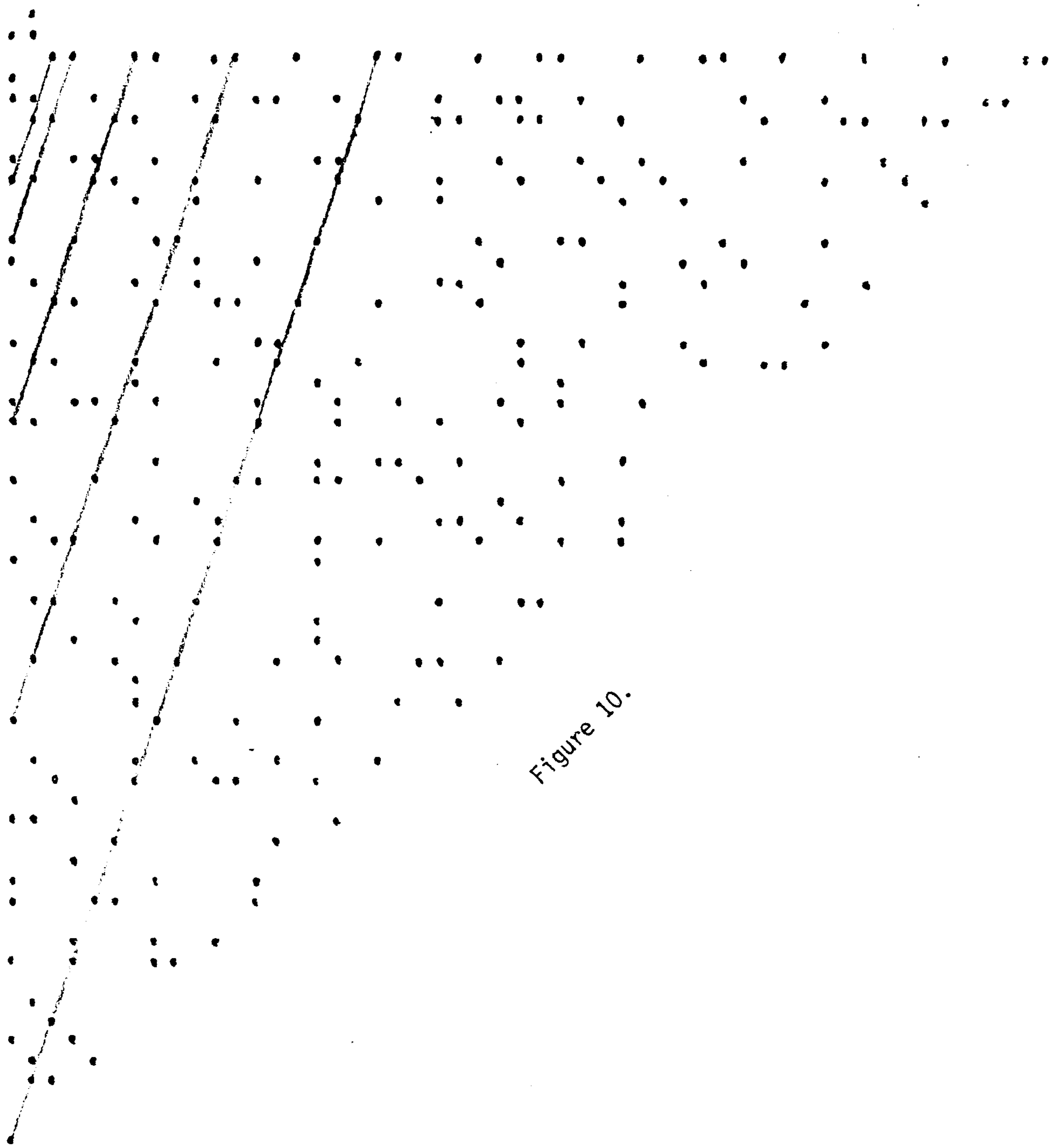

Figure 10.

rent "left-justified" form:

```
1
2  3
4  5  6
7  8  9  10
   etc.
```

Figure 11.

(See figure 12.)

2. THE FORMULAS FOR THE FIVE LINES

Now, what about these straight lines? What algebraic expressions generate them? Indeed, were they actual or had I somehow erred in placing my dots?

After verifying that each dot forming a link in those chains (so to speak) truly represents a prime number, I determined the expression generating each such chain. (See figure 13.)

As an aside, I might point out that the distances between the straight lines form a nice progression: 1, 3, 5, 7. (See figure 14.)

Do the expressions still yield primes for greater values of x? What about for smaller values of x?

While above the upper limits of x given in figure 13 the formulas only haphazardly produce primes, each expression yields a prime for every whole value of x down to zero from the earlier lower limits of x.

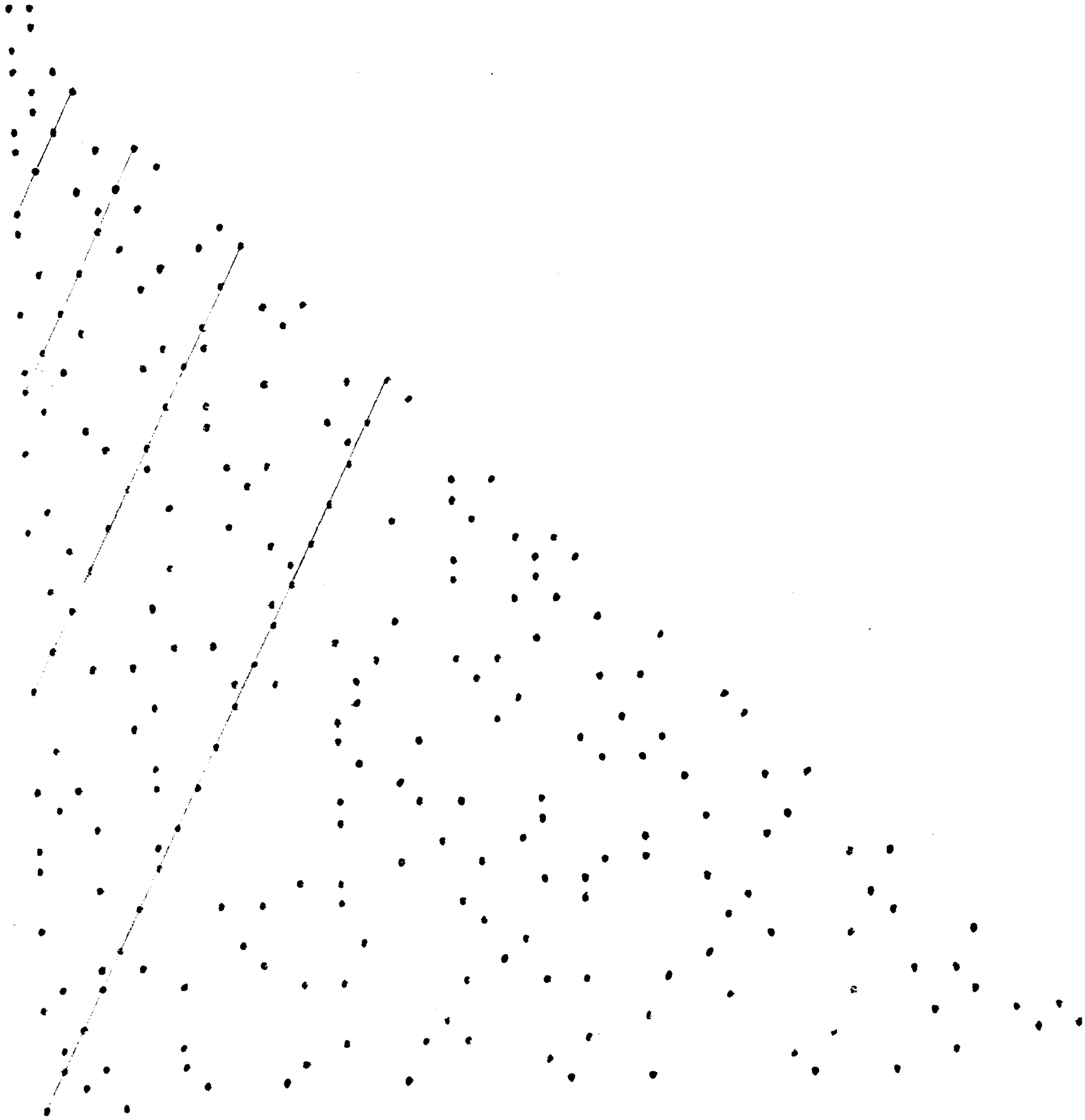

Figure 12.

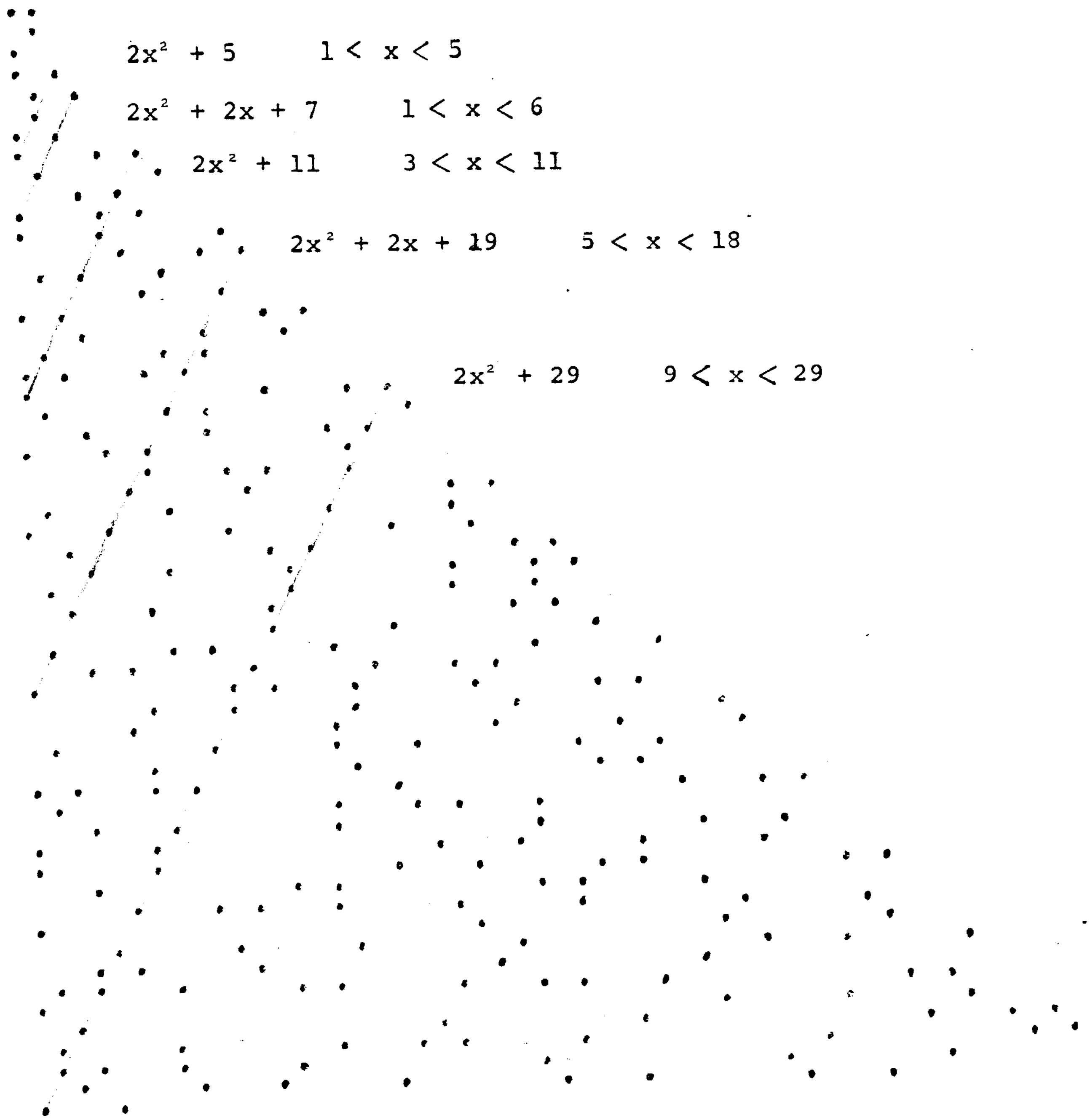
$2x^2 + 5 \quad 1 < x < 5$
$2x^2 + 2x + 7 \quad 1 < x < 6$
$2x^2 + 11 \quad 3 < x < 11$
$2x^2 + 2x + 19 \quad 5 < x < 18$
$2x^2 + 29 \quad 9 < x < 29$

Figure 13.

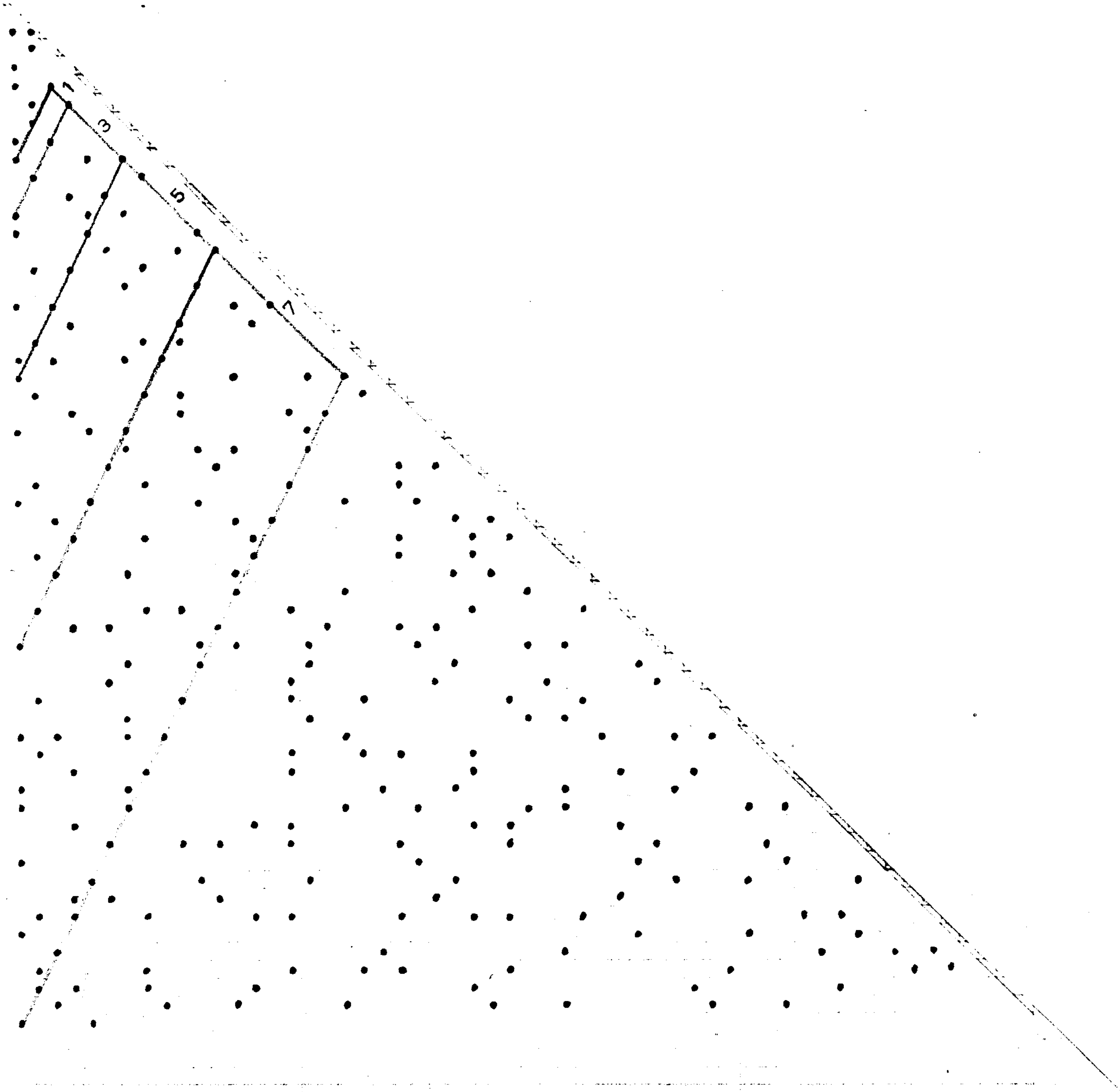

Figure 14.

Thus, the updated ranges for x are:

for $2x^2 + 5$	$-1 < x < 5$
for $2x^2 + 2x + 7$	$-1 < x < 6$
for $2x^2 + 11$	$-1 < x < 11$
for $2x^2 + 2x + 19$	$-1 < x < 18$
for $2x^2 + 29$	$-1 < x < 29$

x whole.

Figure 15.

Since all of the prime sequences produced by these five formulas begin with nice small numbers, it was only natural to inspect the sequence of primes in general for any other simple sequences. And, sure enough, there are more simple sequences.

First I found one starting with 17: 17, 19, 23, 29, 37, 47, etc. up to 257. Then I found another, starting with 41: 41, 43, 47, 53, 61, 71, etc. up to 1601. This was by far the most remarkable sequence, holding true for forty consecutive values! The formula for this sequence is

$$x^2 + x + 41$$

while that for the sequence starting with 17 is

$$x^2 + x + 17.$$

Their valid ranges of x are: $-1 < x < 40$ and $-1 < x < 16$ respectively.

These two formulas bear a striking resemblance to certain of the original five formulas found from the triangular array, those of form $2x^2 + 2x + c$ where c is a constant like 7 or 19. Perhaps the two new formulas, along with the initial five, formed part of some larger, all-embracing set of formulas?

I couldn't find any more $x^2 + x + c$ type sequences for values of c higher than 41, but there are two more such sequences for values of c smaller than 17:

$$x^2 + x + 5 \qquad -1 < x < 4$$

and $$x^2 + x + 11 \qquad -1 < x < 10.$$

Figure 16.

3. THE FAMILY OF FORMULAS

I now had nine formulas. Listing them in order by their constant, I settled on the following arrangement:

$$x^2 + x + 5$$

$$2x^2 + 5$$

$$2x^2 + 2x + 7$$

$$x^2 + x + 11$$

$$2x^2 + 11$$

$$x^2 + x + 17$$

$$2x^2 + 2x + 19$$

$$2x^2 + 29$$

$$x^2 + x + 41.$$

Figure 17.

There is a fascinating aspect to this list -- the formats of the formulas themselves form a pattern:

$$x^2 + x + c$$

$$2x^2 + c$$

$$2x^2 + 2x + c$$

$$x^2 + x + c$$

$$2x^2 + c$$

$$x^2 + x + c$$

$$2x^2 + 2x + c$$

$$2x^2 + c$$

$$x^2 + x + c.$$

Figure 18.

There is perfect symmetry of form stemming in both directions from the middle formula towards the first and last formulas, respectively. There is a formula of form $x^2 + x + c$ at both ends of the list. The other two $x^2 + x + c$ formulas are the fourth formula in from their respective ends of the list. Besides the $2x^2 + c$ type of formula at the center of the list, the other two of that type are each the second from their respective ends of the list. Finally, the two $2x^2 + 2x + c$ type formulas are respectively the third formula in from their ends of the list. Thus the list can be represented in form by the following symmetrical list:

A
B
C
A
B
A
C
B
A.

Figure 19.

It is this symmetry, this inherent self-containment, that leads me to conjecture that my nine formula family may contain all such formulas in existence. I might add that there are no signs of any more unbroken lines of primes when the triangular array is extended down to 133 rows (down to about the number 9,000). (See figure 20.)

It was at this point in time, 1969, that I contacted a professor specializing in number theory, a Dr. Whiteman at U. S. C., and reported my findings over the phone. He was delighted, but he somewhat tarnished my pride of discovery by revealing that $x^2 + x + 41$ was well-known and that the mathematician Euler had extensively studied "prime-rich" formulas 200 years earlier. (While writing this paper I decided to try to look up just what Euler had known. I visited the Fine Mathematics Library at Princeton University and looked up everything I could find on Euler. There was very little written in English, and nothing even remotely resembled "my" nine formulas. I then resorted to the painful task of scanning Euler's collected works -- running over twenty thick volumes -- for any sign of the formulas. The reason that that scanning was painful was that none of the writings are in English -- most are in Latin, some are in French, and some bits are in Euler's native German. I could find nothing relevant though there were chapters dealing with prime number theory. I suppose I just didn't hit the right spots.)

Subsequent to my conversation with Dr. Whiteman I sheepishly discovered that not only is $x^2 + x + 41$ mentioned in many written discussions of primes, but in addition both it and $x^2 + x + 17$ are cited even in the very article in that March, 1964 issue of <u>Scientific American</u> that had launched me on this involved quest.

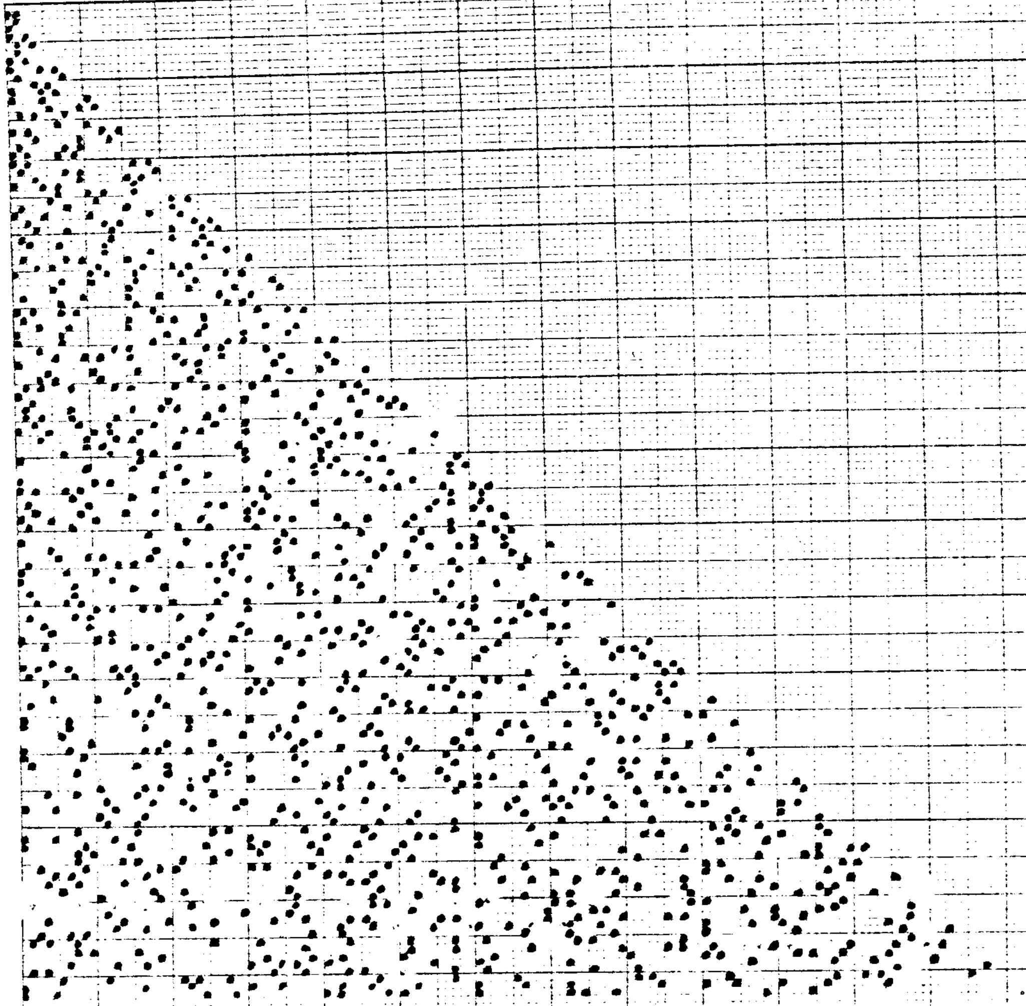

Figure 20.

4. $X^2 + X + 41$ ON THE TRIANGULAR ARRAY

Undaunted, I continued my studies of primes and patterns. I asked myself what $x^2 + x + 41$ would look like plotted on the triangular array. Since it wasn't a straight line, what would it be? (See figure 21.) As you can see, the dots of $x^2 + x + 41$ do tend to form fragments of curves.

5. EXTENSION OUT OF THE TRIANGLE

I decided to extend those curve fragments outside the boundary of the "hypotenuse" of the triangular array (See figure 22.) -- wow!

Each dot now seemed to be a member of several different curves, each curve going in a different direction. And, there seemed to be some kind of convergence point at the top center of the graph, where all the curves come together. It was this convergence effect that was to shortly lead me to explore this snail-shell shaped set of curves beyond even these confines. But before checking, I first wanted to know what $x^2 + x + 17$ would do on the triangular array.

It turned out that $x^2 + x + 17$ produces the exact same curves, only displaced to the right from the left edge of the graph by 17 positions instead of 41. It was an easy step from there to realizing that I could shift the convergence point all the way to the left edge of the graph by eliminating the constant altogether from $x^2 + x + c$ (since c dictates the amount of displacement to the right from the left edge of the graph). So, just plain $x^2 + x$ would produce the dazzling dot curves.

6. "THE DESIGN"

The next step was to extend the curves above the top of the graph and see just what the convergence point looked like from all sides. (See figure 23.) PHEW!

7. THE CURVE FAMILIES AROUND THE AXIS OF SYMMETRY

Looking at "the design" with the void at the bottom there is an obvious left and right symmetry. Further, there appear to be distinct families of curves composed of the dots of "the design". For example, extending from the vertex of the void straight upward is one family of narrow, nested, downward curving curves. (See figure 24.)

Then one can see both immediately to the left and right of that central family two slightly wider downward curving curve families. (See figure 25.)

Then, at the far left and right edges of "the design" are two broad downward curving curve families. (See figure 26.)

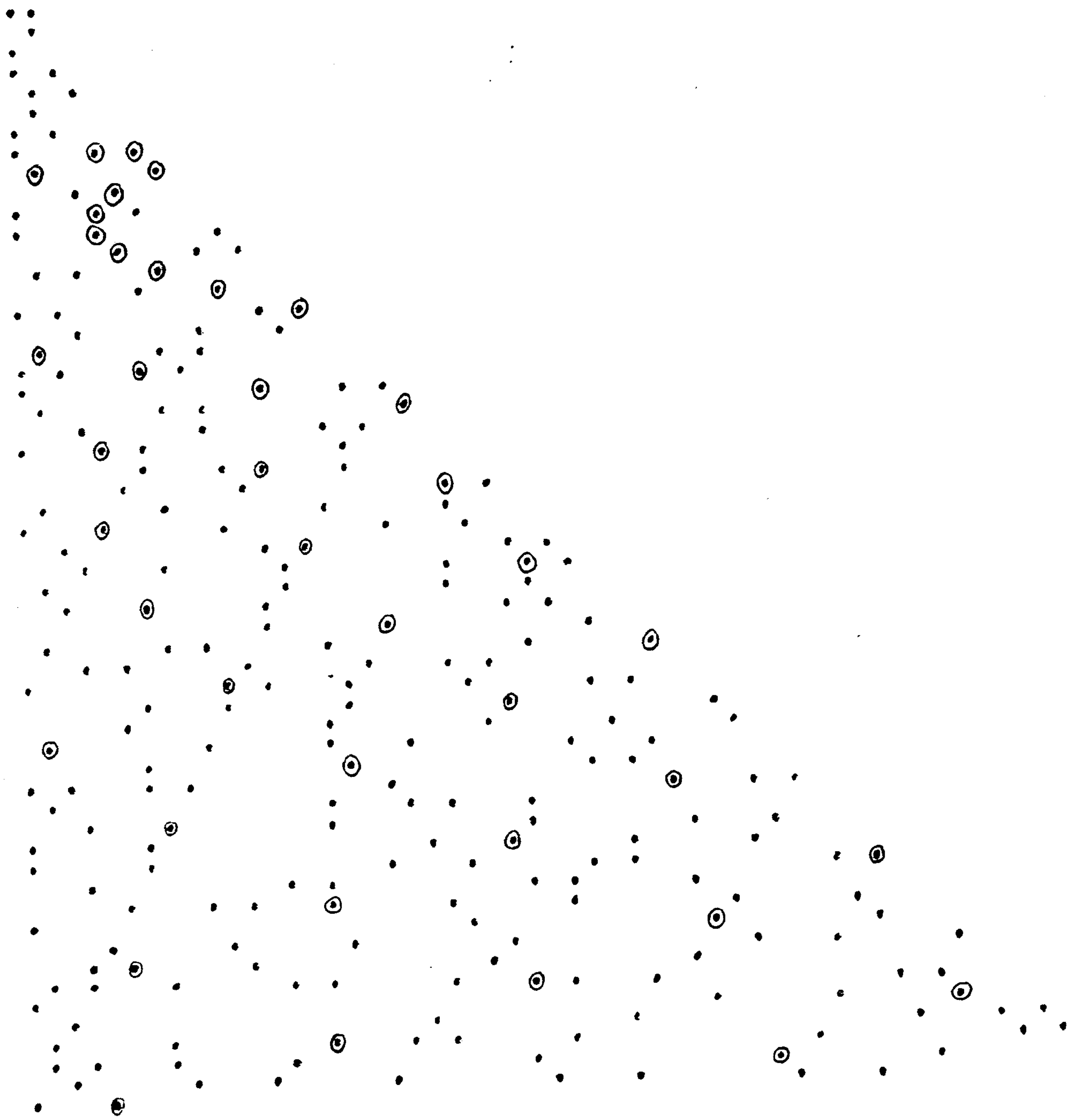

Figure 21.

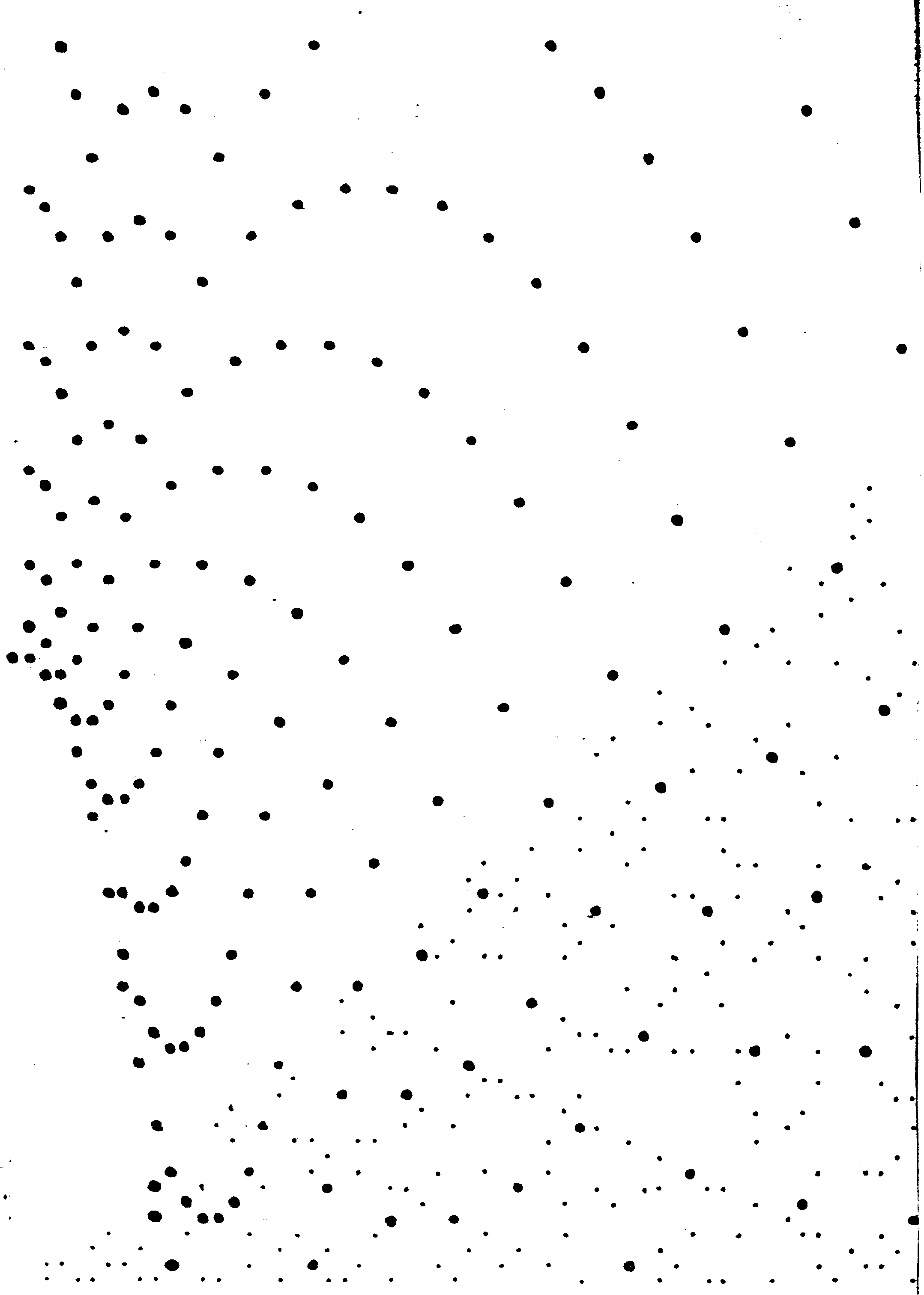

Figure 22.

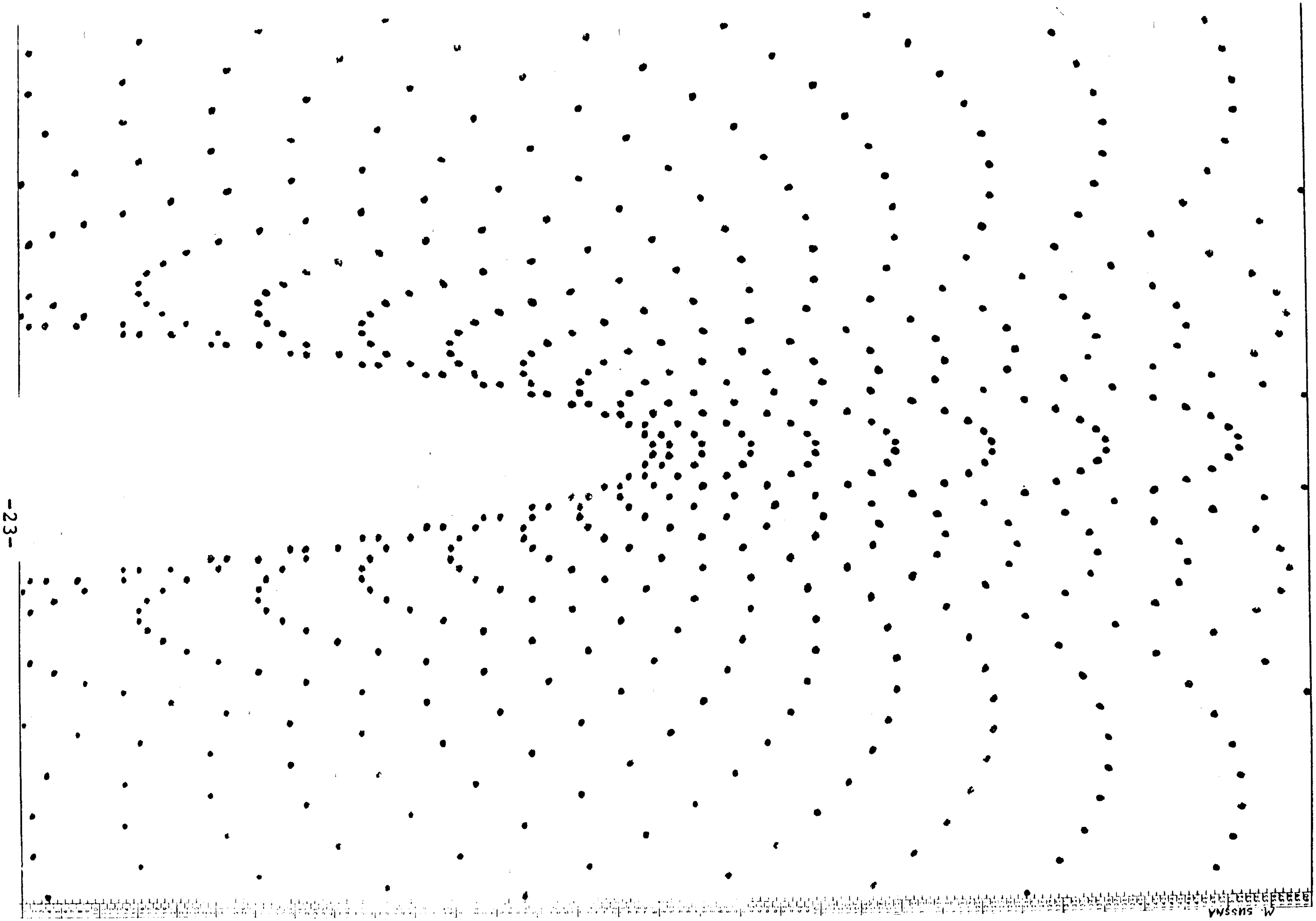

Figure 23.

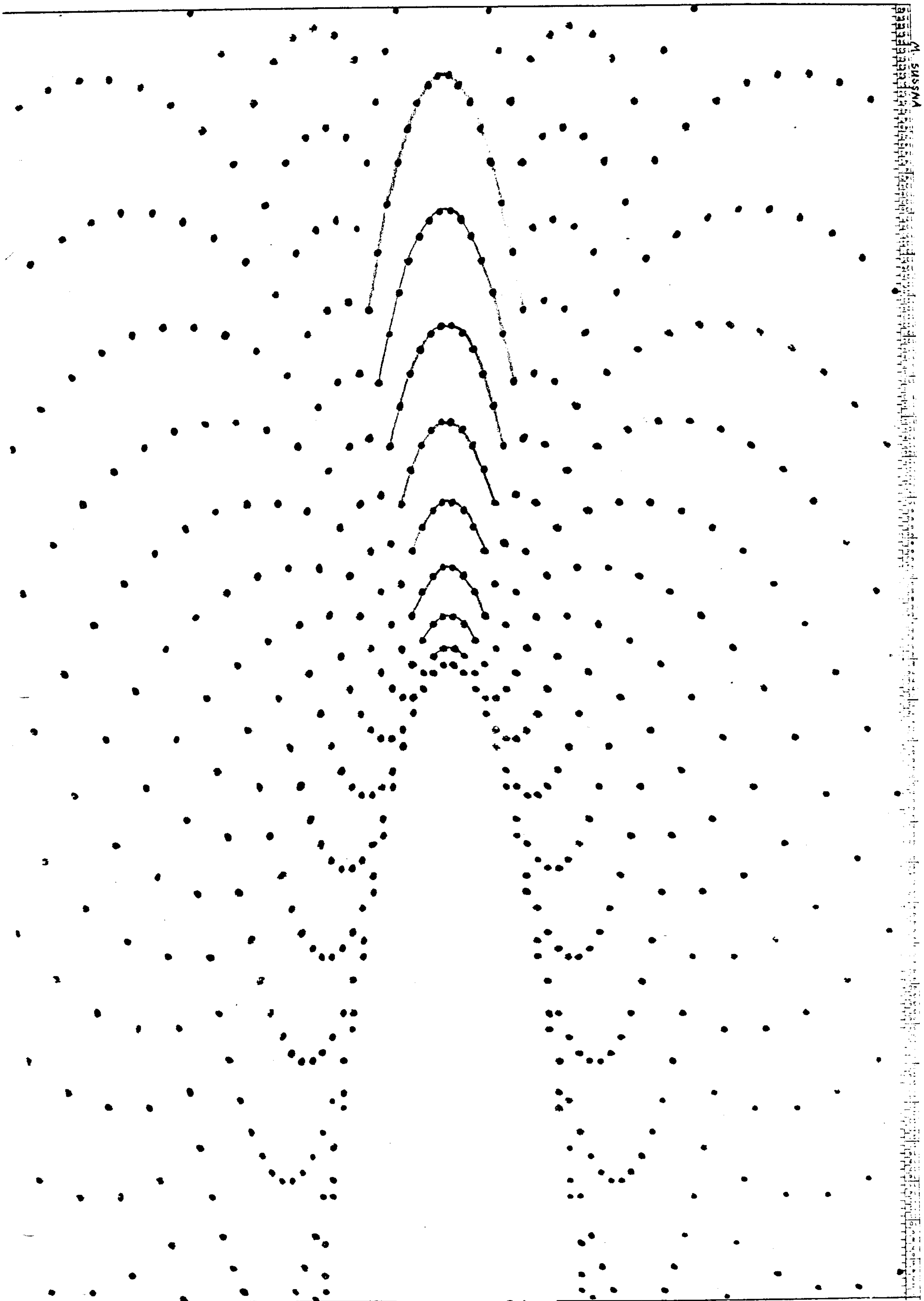

Figure 24.

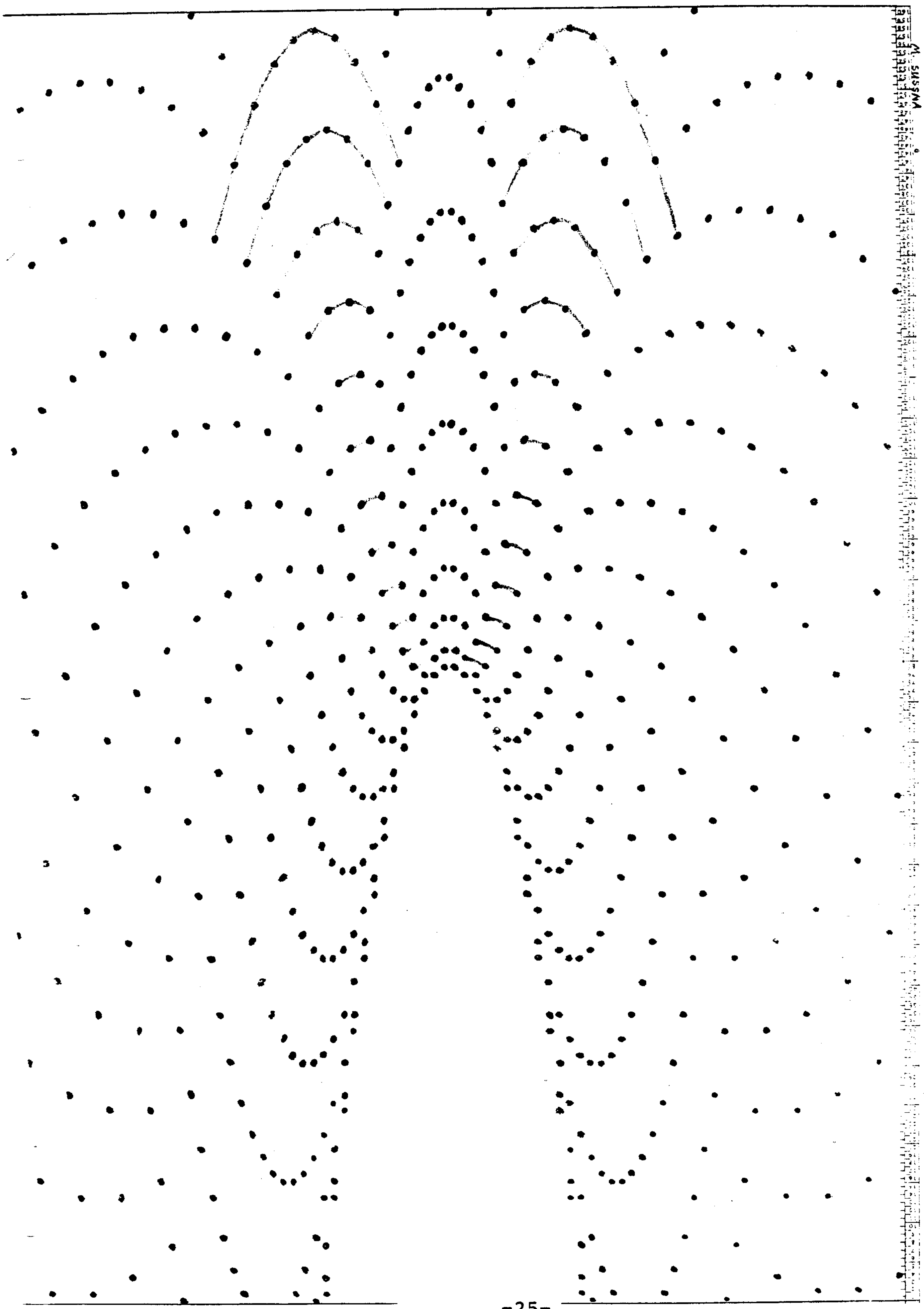

Figure 25.

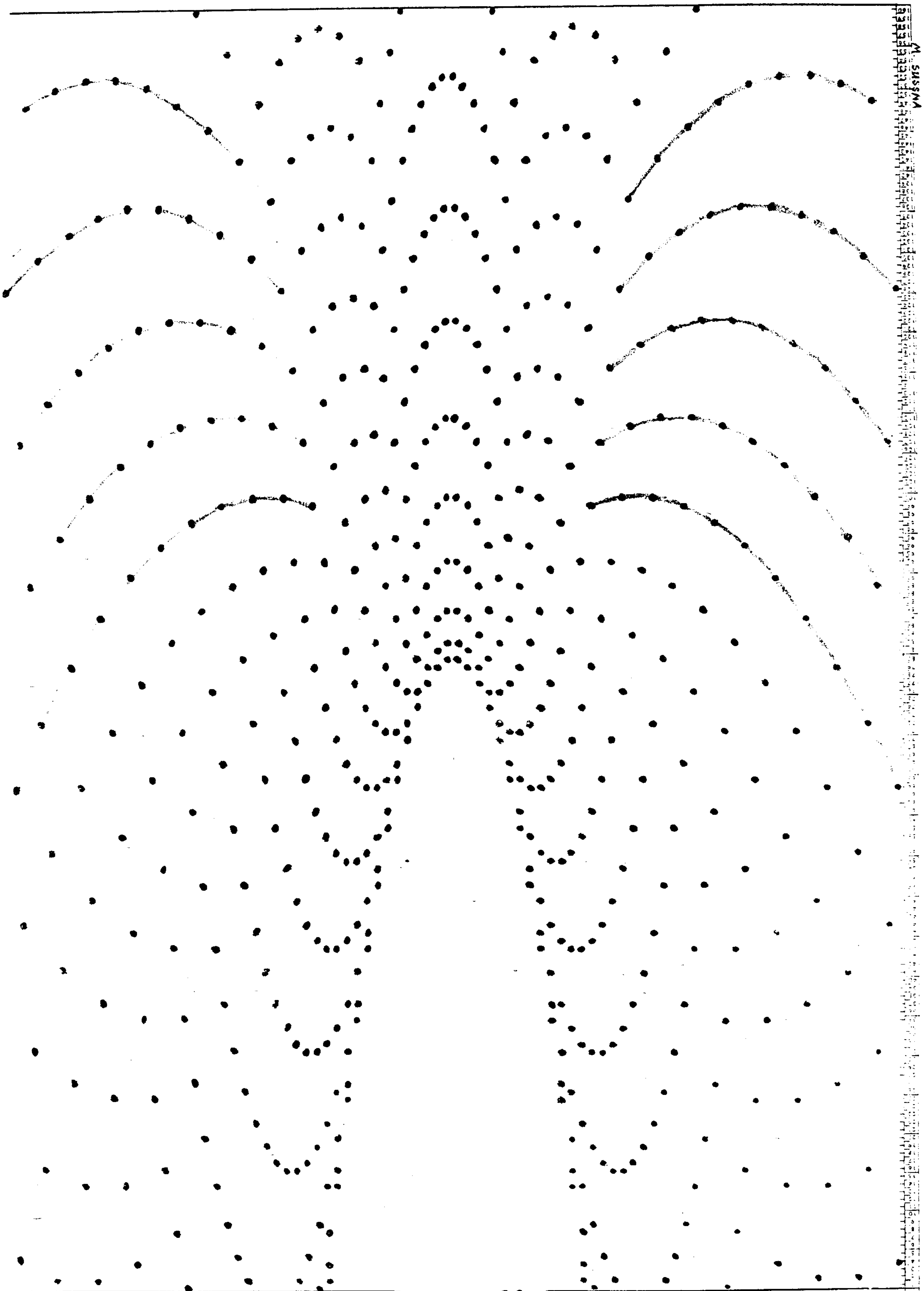

Figure 26.

This trend toward wider families no doubt continues on both sides indefinitely.

Now, one may also note upward curving narrow curve families immediately adjacent to the void, on both its left and right. (See figure 27.)

And again, there are broader upward curving curve families to the left and right of the two void-adjacent families. (See figure 28.)

Here, too, the trend will continue outward on both sides.

In case anyone is interested, the formulas for the five above mentioned curve families' (left and right being considered equivalent) individual member curves when viewed as opening upward (for simplicity's sake) are:

curve family	individual member
central	$1/2x^2 - 1/8$
left and right center	$1/4x^2$
far left and right	$1/18x^2 - 1/8$
void adjacent	$1/2x^2 - 1/8$
outer void adjacent	$1/16x^2 - 1/4$

Figure 29.

Another note on the curve families: "the design" seems to be divided into two halves not only down the middle when the void is at the bottom, but left to right at the vertex of the void, as well. (See figure 30.)

No downward curving curves have their vertex below the horizontal dividing line at the vertex of the void, while no upward curving curves have their vertex above that dividing line.

When "the design" is turned so that the void is either at the left or right, and viewed on edge, another surprise awaits us. There are completely empty swaths across the whole graph, sometimes more than one column wide. (See figure 31.)

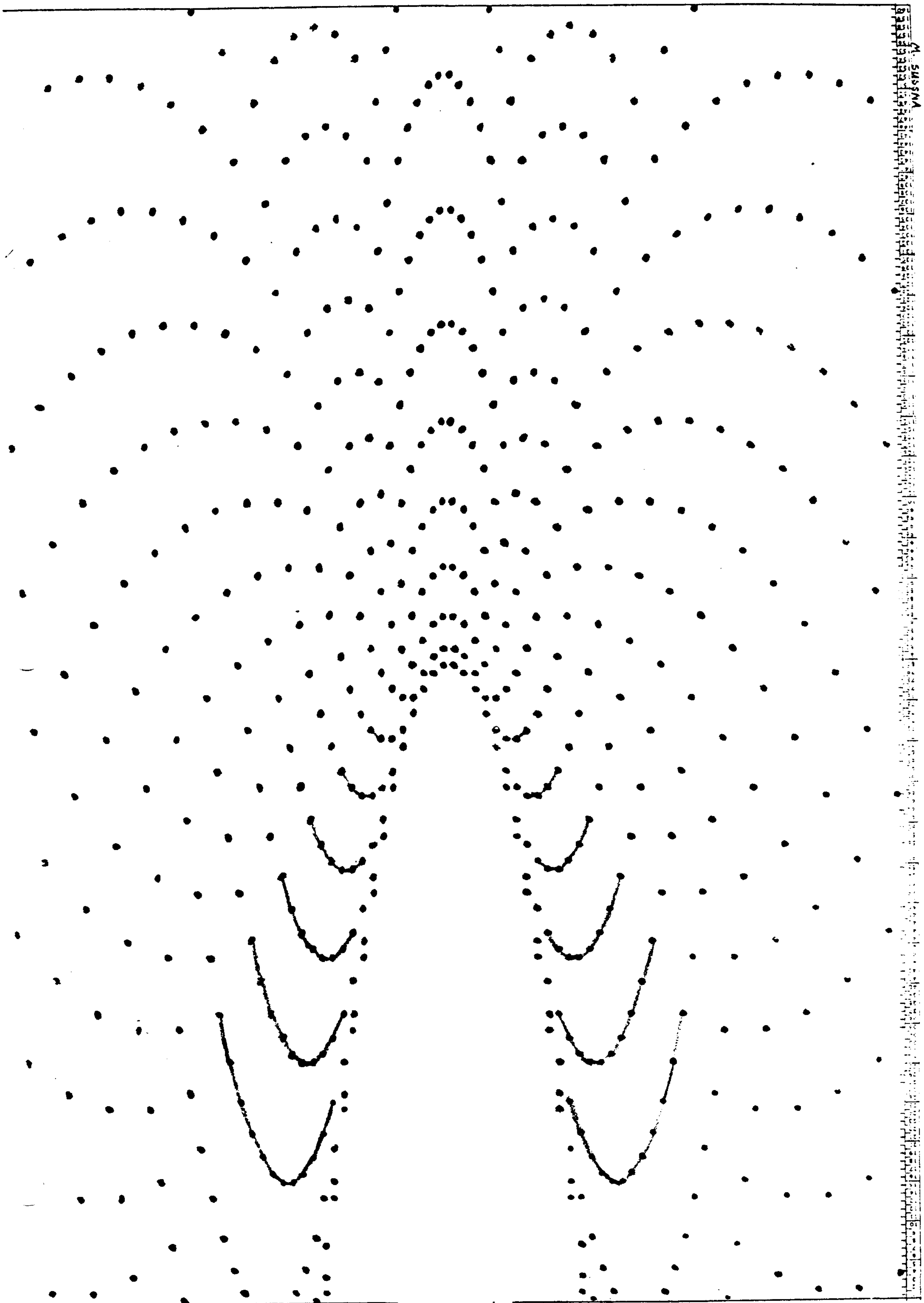

Figure 27.

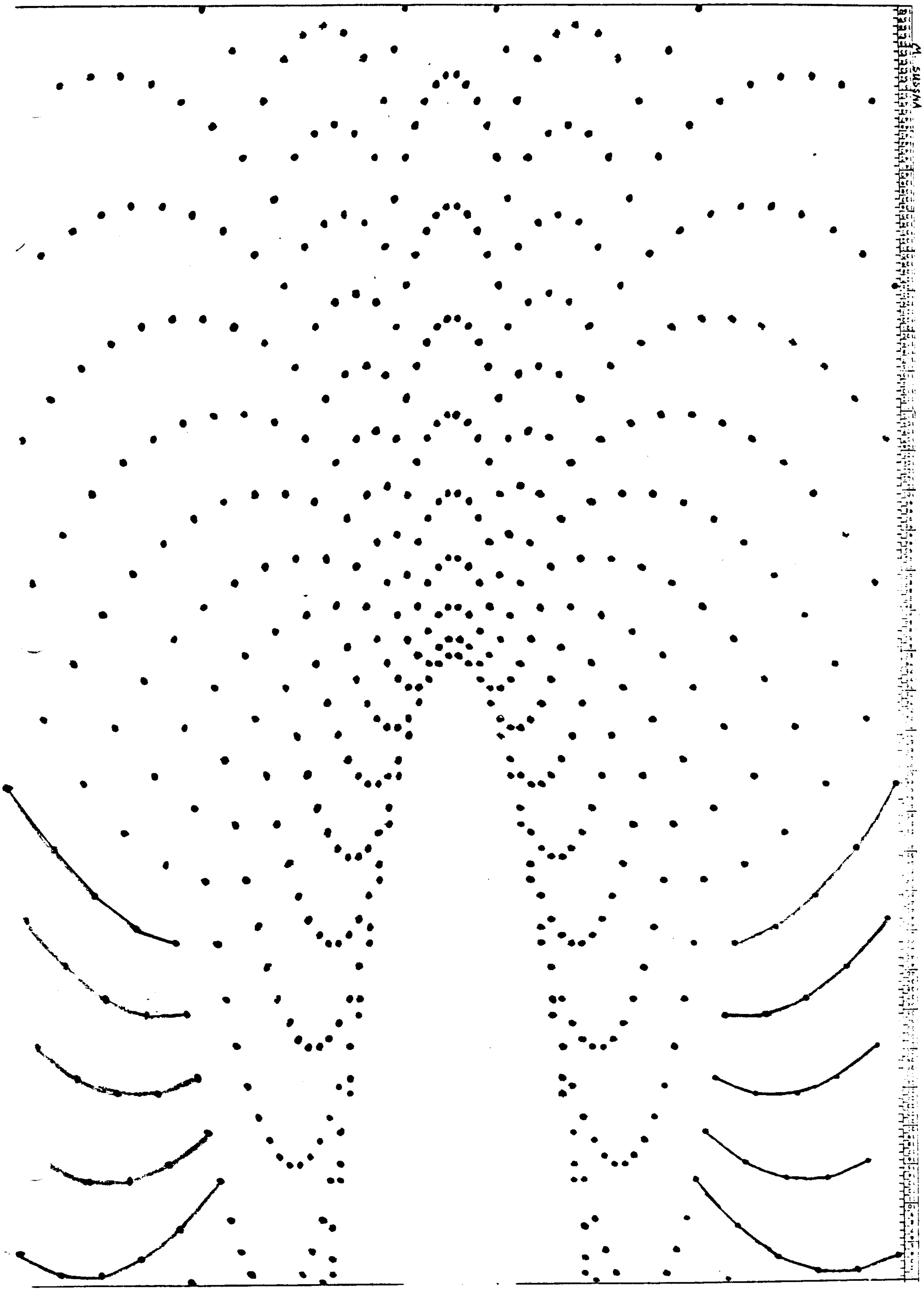

Figure 28.

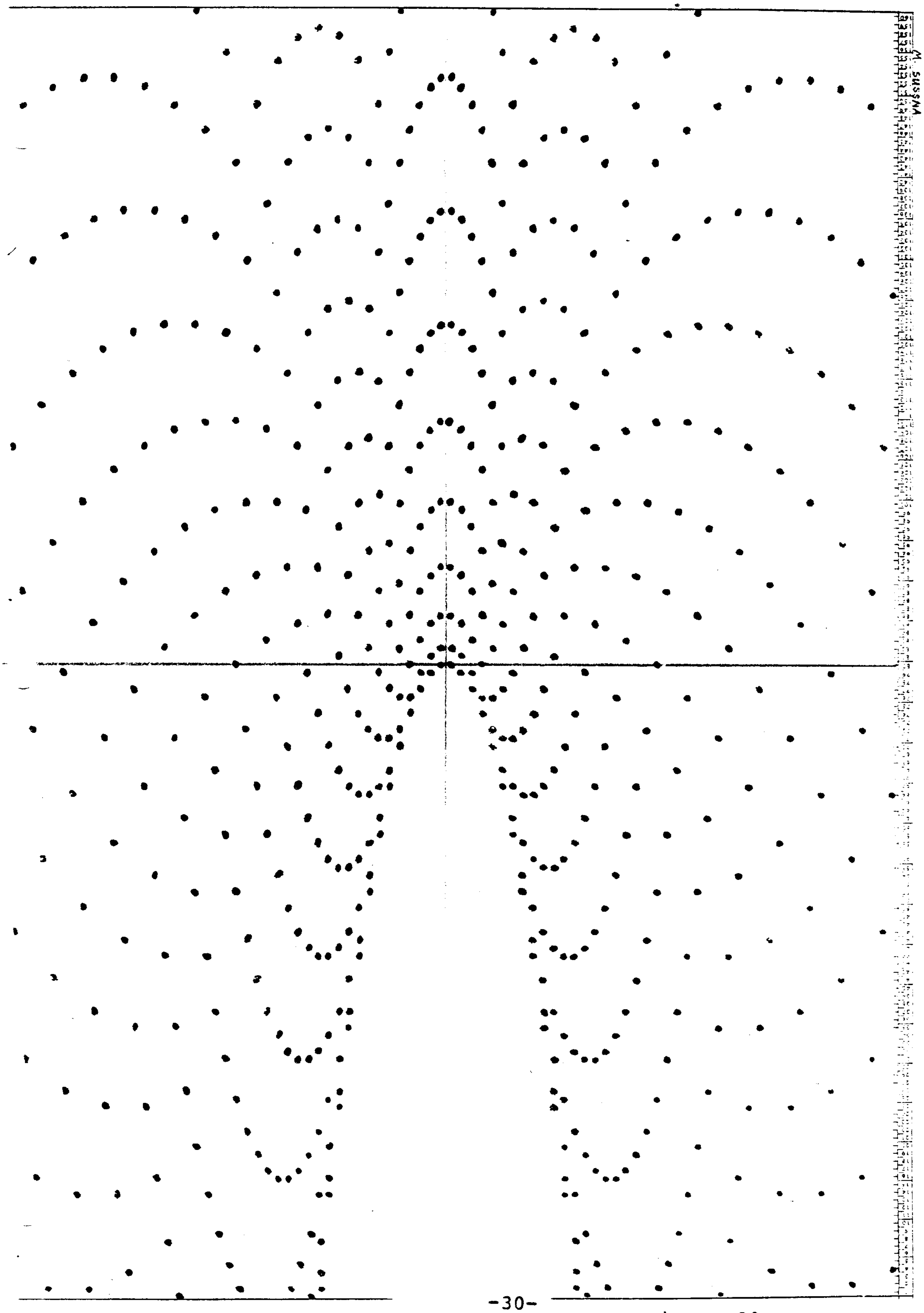

Figure 30.

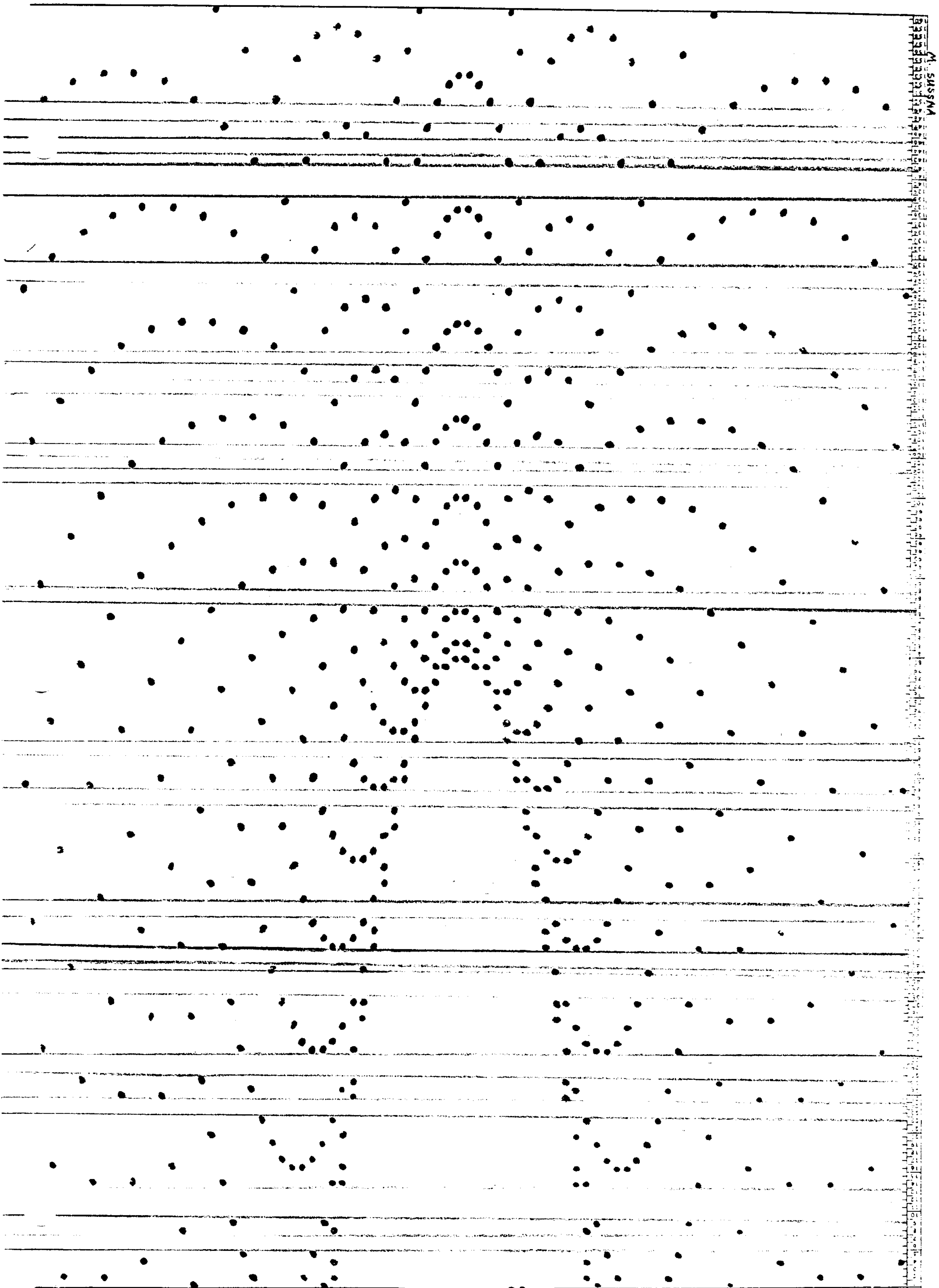

Figure 31.

8. "THE DESIGN" ON EDGE

Look at "the design" from the void's open end, on edge.

The left and right void adjacent families now extend all the way to the other end of the paper and all the way to each side of the paper.

The far left and right void adjacent families have disappeared -- so have the left and right center downward curving families and the far left and right downward curving families.

The only families remaining are the left and right void adjacent families and the central downward curving family. (See figure 32.)

When "the design" is viewed in the same manner with the void at the top, one realizes that the entire "design" is one family! (See figure 33.)

9. GENERATING THE WHOLE "DESIGN" FROM ITS PARTS

Thus, one of the most remarkable features of "the design" is that the entire "design" can be generated from one or another of its parts alone.

The central downward curving curve family can individually produce "the design", while each left and right <u>pair</u> of families, whether upward or downward curving, can do the same. Words don't suffice to tell you -- let me show you:

Take the central downward curving curve family. Start first with a single member of the family -- say the one with its vertex at the vertex of the void. (See figure 34.)

Now extend the family by adding members above the first member. Each succeeding member is directly above and the exact same shape as each of its predecessors. If the first curve, the one with its vertex at the vertex of the void, has its vertex at y=0, then the next curve above it has its vertex at y=2, the curve above that has its vertex at y=6, and so on. In general, the central downward curving curve z has its vertex at $y = z^2 + z$, where z=0 for the curve whose vertex coincides with that of the void. (See figure 35.)

This adding on of curves has an intriguing effect as it procedes: the more curves that are added on, the more "design" that is finished. The "completing" moves out in every direction from the center of "the design", the vertex of the void. (This is the old "convergence point" whose lure drew me out past the confines of the graph containing the triangular array.) As the number of curves grows in-

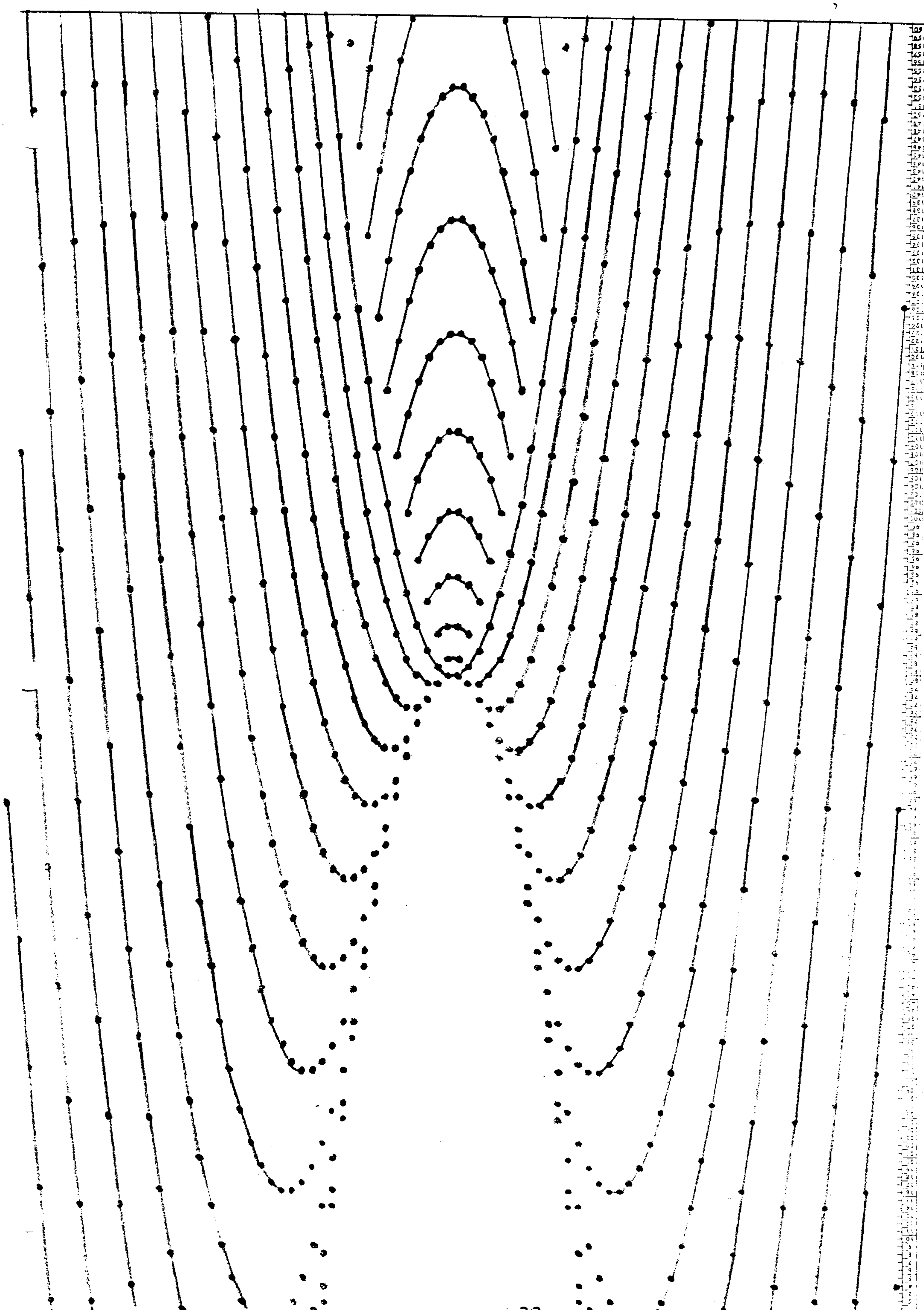

Figure 32.

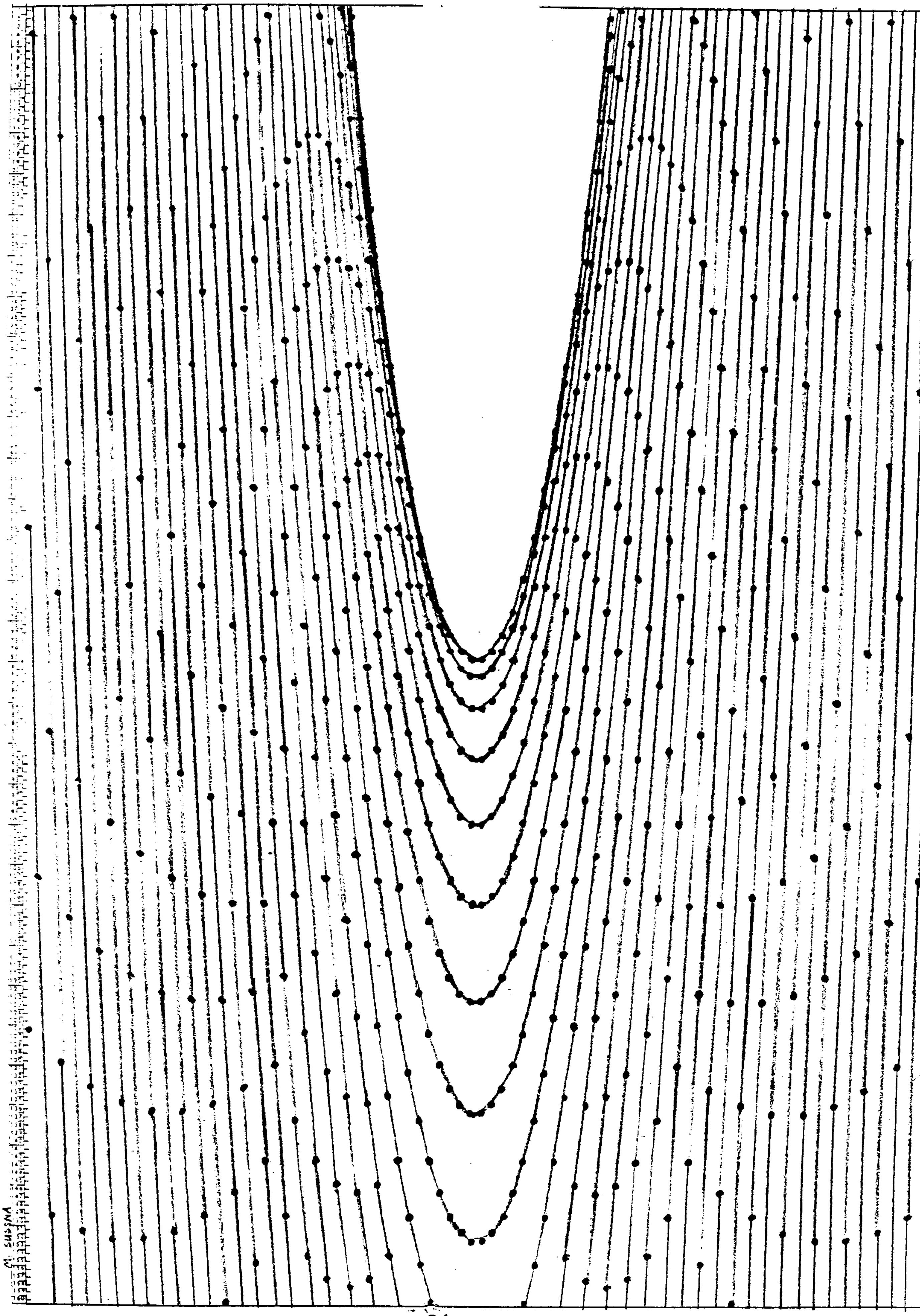

Figure 33.

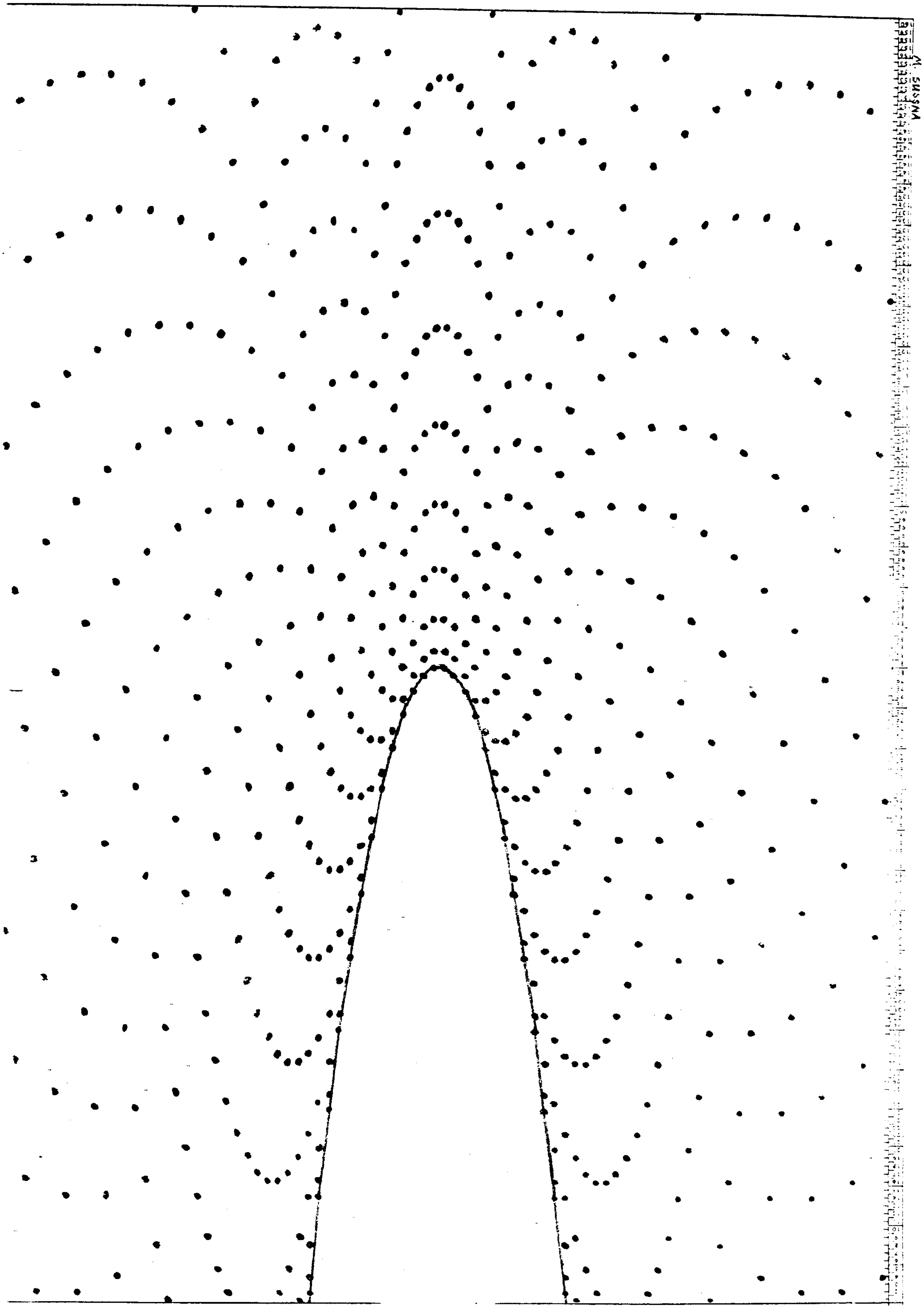

Figure 34.

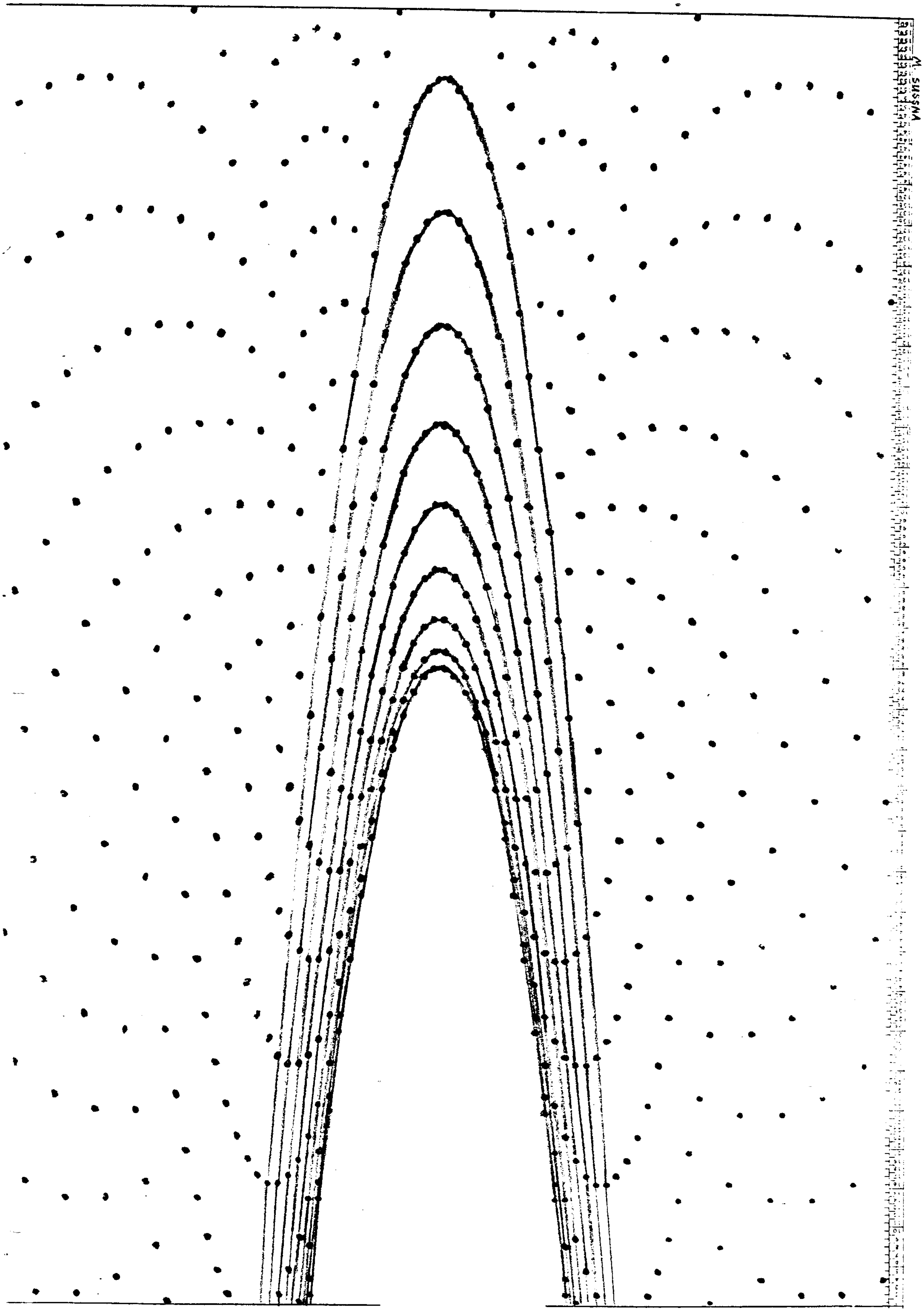

Figure 35.

definitely great, the area of "design" also grows, almost as greatly. For example, only about 32 curves of the central downward curving curve family are needed in order to generate the entire "design" as presented here. (See figure 36.) In "the design" presented here, the edge of the paper at the wider end of the void is an arbitrary truncation of all the central downward curving curves that intersect it. In fact, all central downward curving curves intersect it. Of course the same is true of all four sides of any rectangular representation of a central portion of "THE DESIGN": they all truncate the "true" "DESIGN" - that is, the true "DESIGN" goes on endlessly in every direction. Thus any of our attempts to render it produce necessarily limited pictures of it.

Taken together, the left central downward curving curve family and the right central will produce the entire "design" in much the same way, and so on for any left and right downward or upward curving curve family pair. (See figures 37-40.)

One thing this generation of the whole from any of its parts process reminds me of is the biological regeneration of an organism from one limb, as in the starfish, or even from a single cell, as in cloning.

"The design" has elements of multiplicity in its many curve families. It has elements of duality in its bilateral symmetry and the pairs of curve families. It also has elements of unity in that a single curve family can generate the whole "design" and that each dot is a member of all curve families. The multiplicity of its parts is complemented by the unity of the whole. In other words, although from one point of view (looking at it from above) there appear to be many subsections to the "design" (the curve families), from another perspective (looking at it on edge) there is only one curve family. Which viewpoint is correct? Both are!

Thus, philosophically speaking, the one and many each is illusory from the perspective of the other, but the truth is that both are valid.

10. DOTTED LINES TO THE NORTHEAST

Some of my more recent findings include the following.

The values of the $2x^2$ formulas, as you will recall, appear as straight lines of primes on the triangular array. Early on I had asked myself what happened with the values when x is between zero and the number generating the first value in a given straight line. At that time I had simply plotted those lower values on the triangular array. They formed curves as seen in figures 41-45.

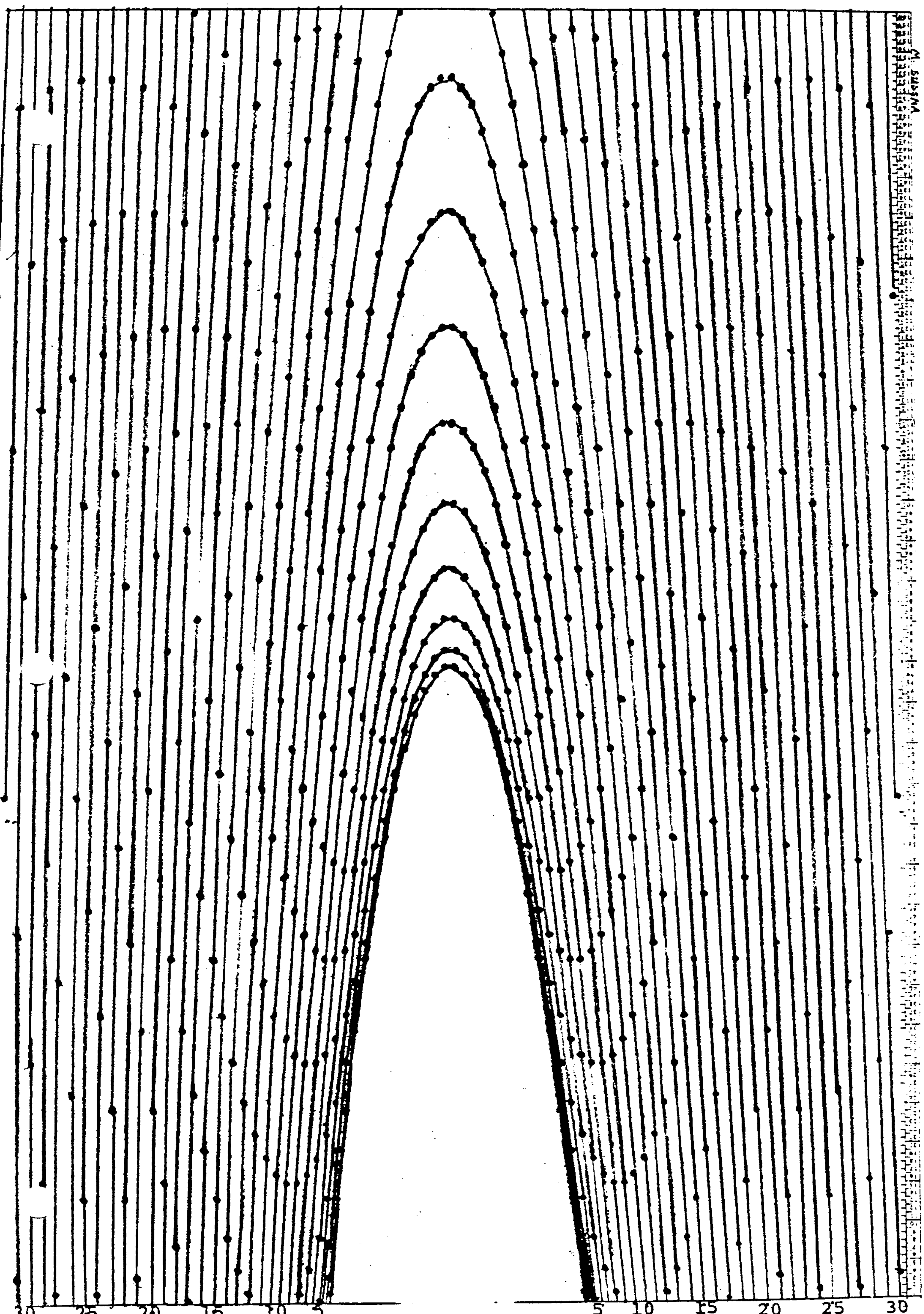

Figure 36.

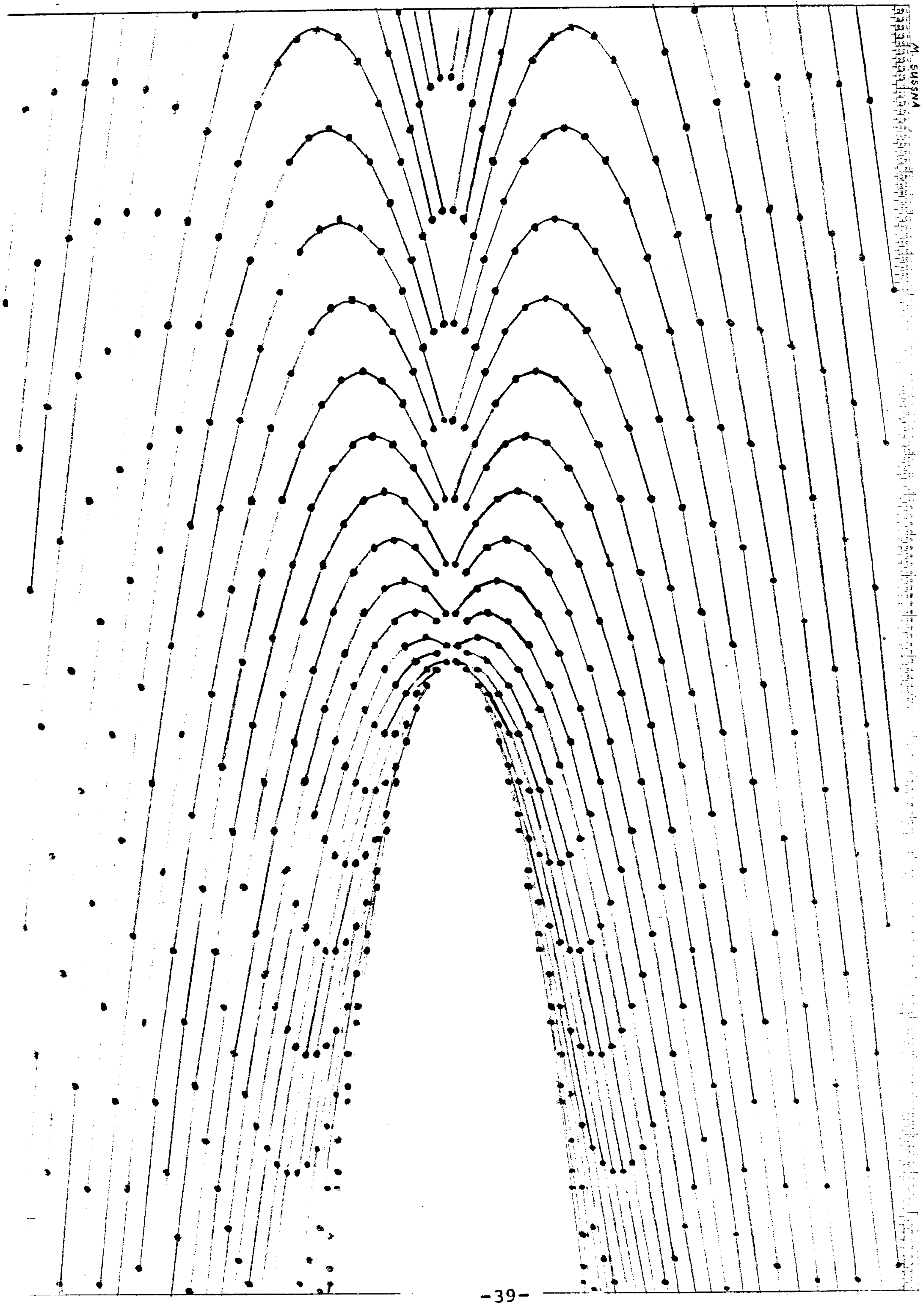

Figure 37.

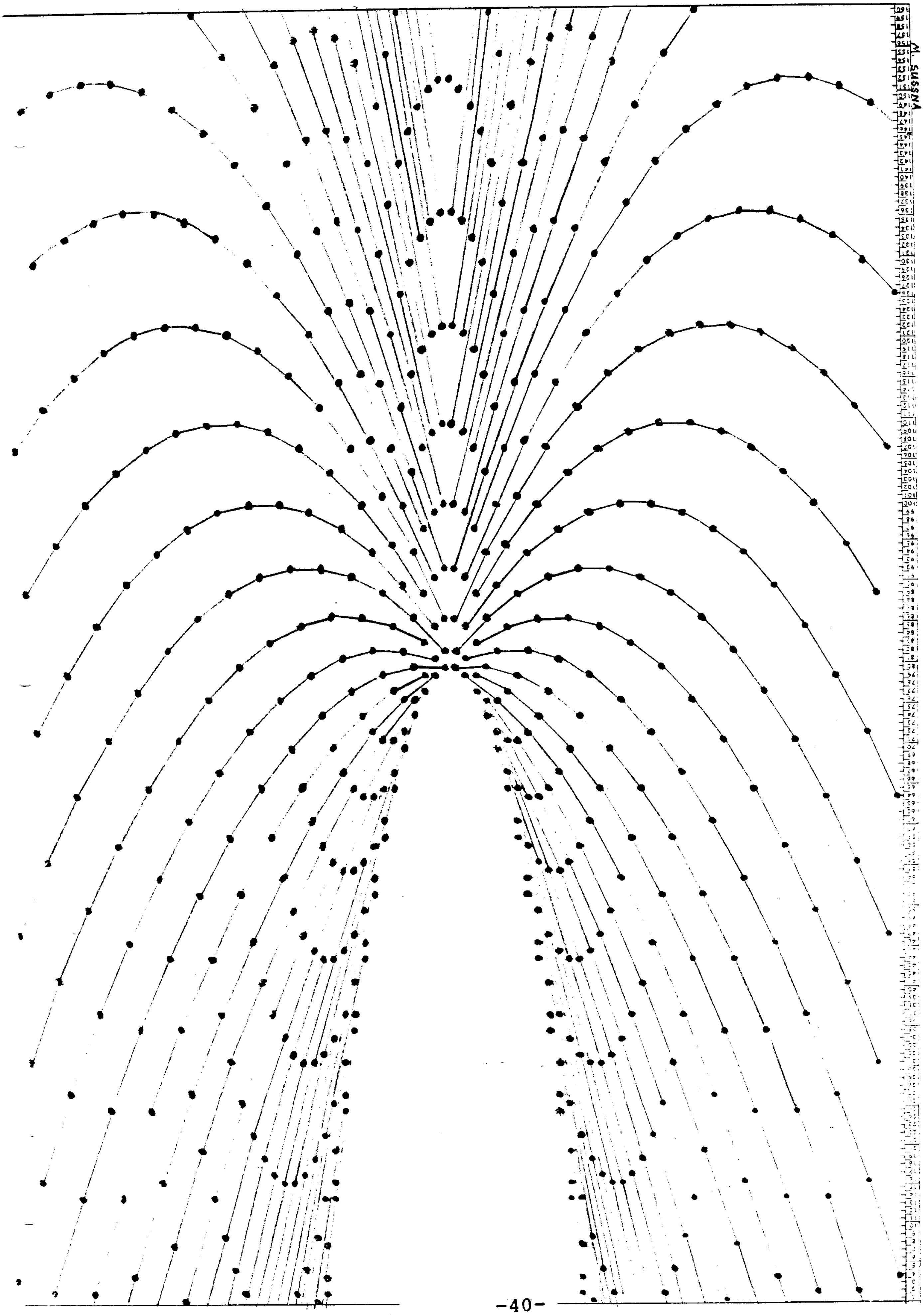

Figure 38.

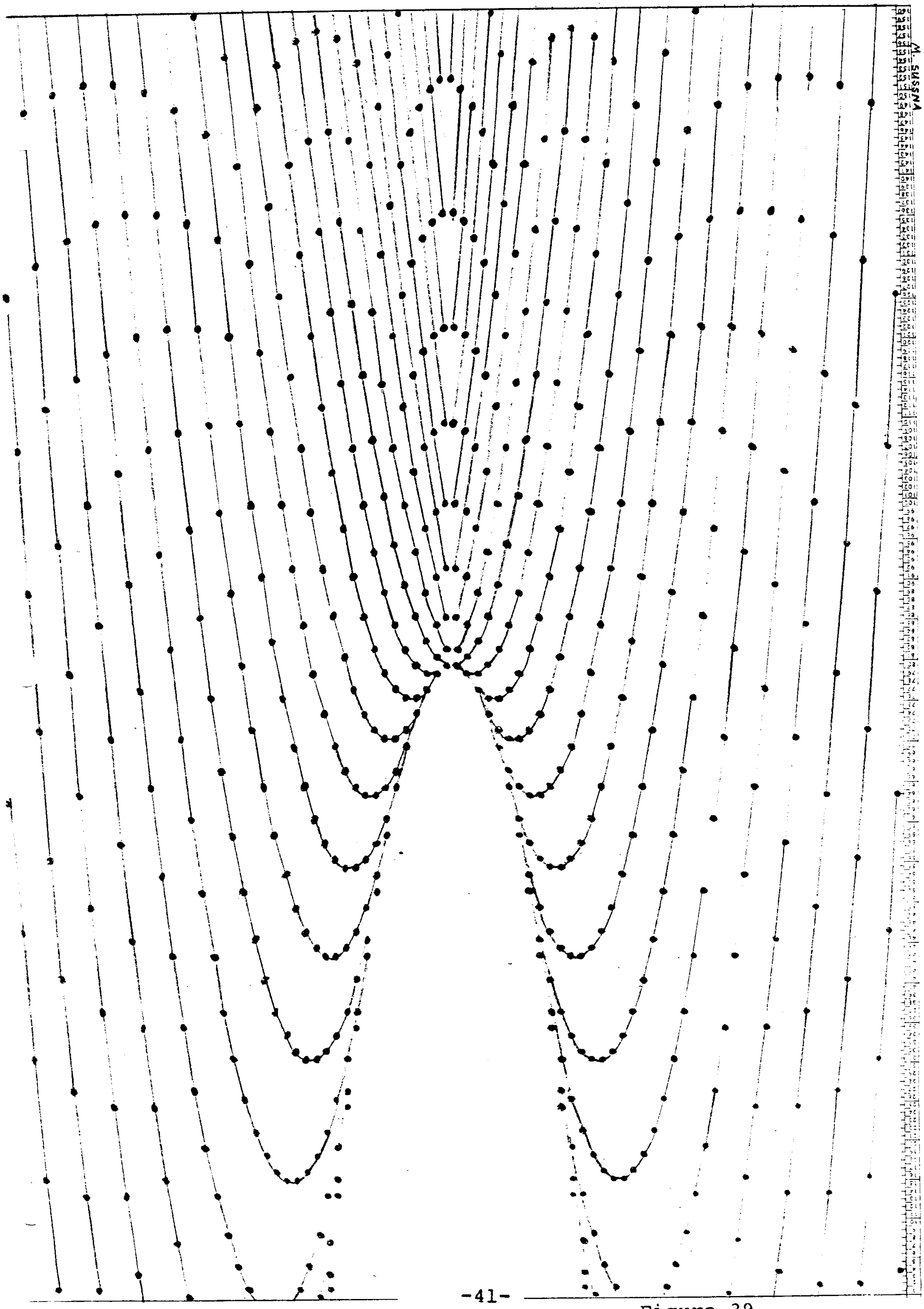

Figure 39.

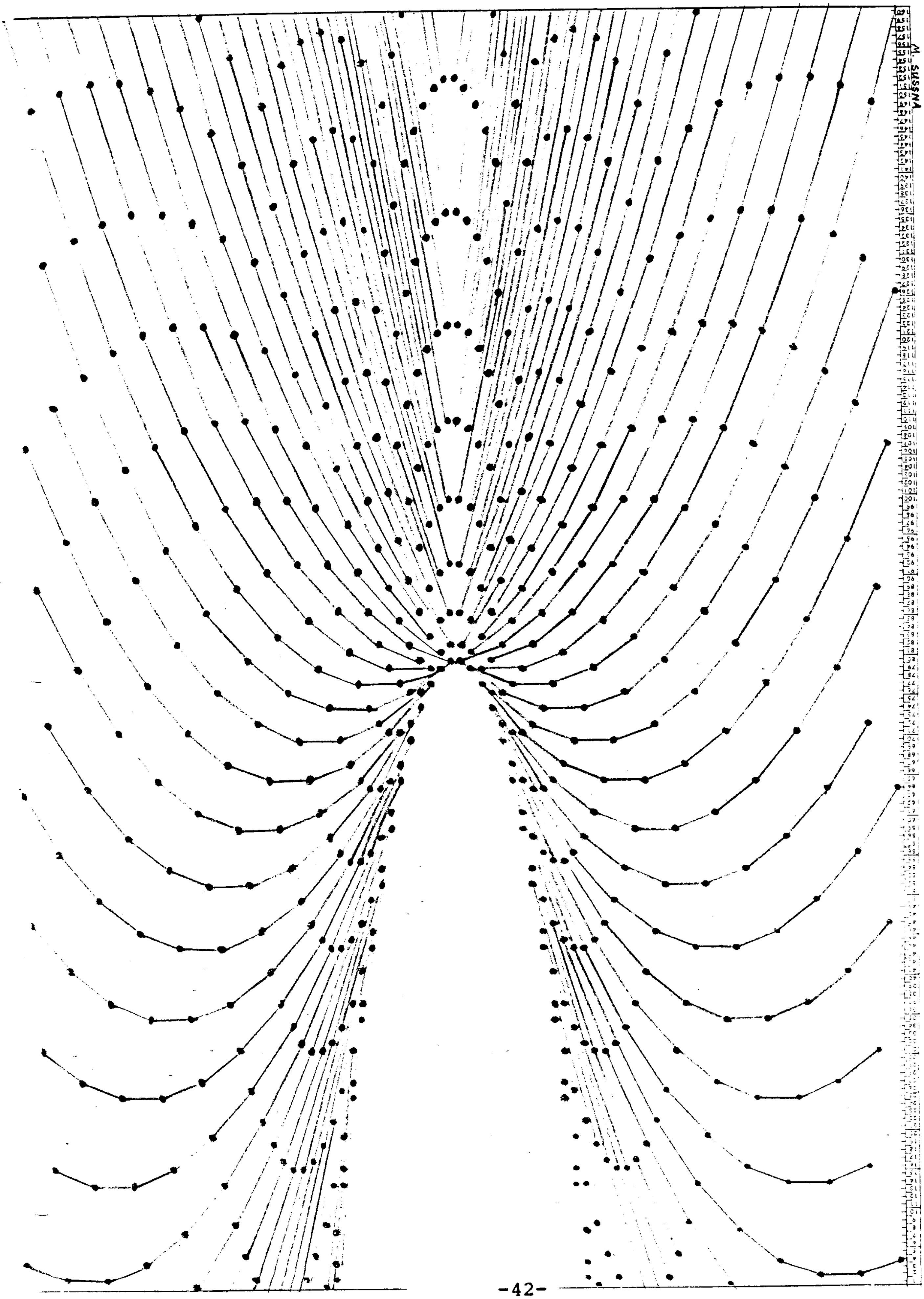

Figure 40.

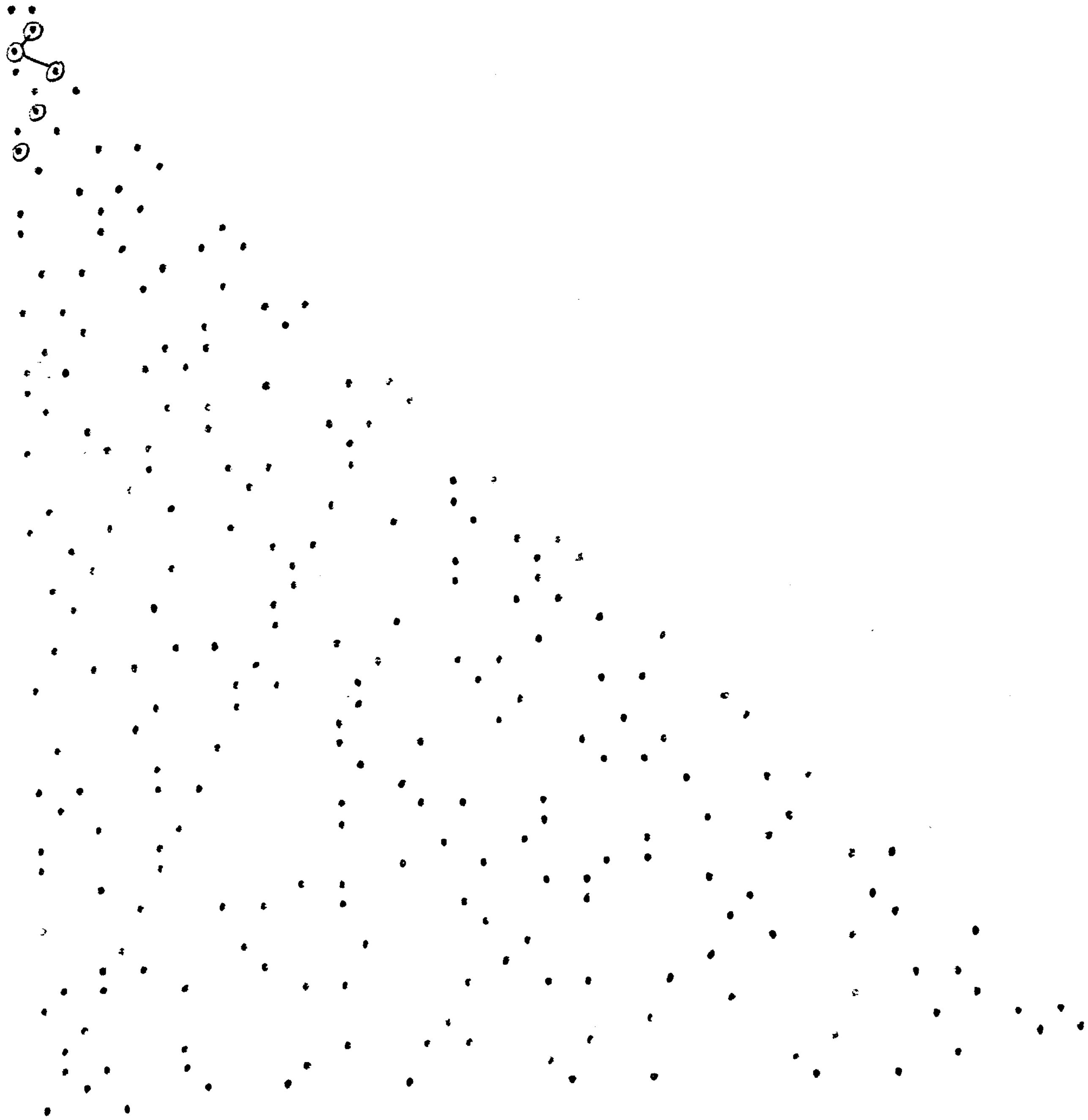

Figure 41.

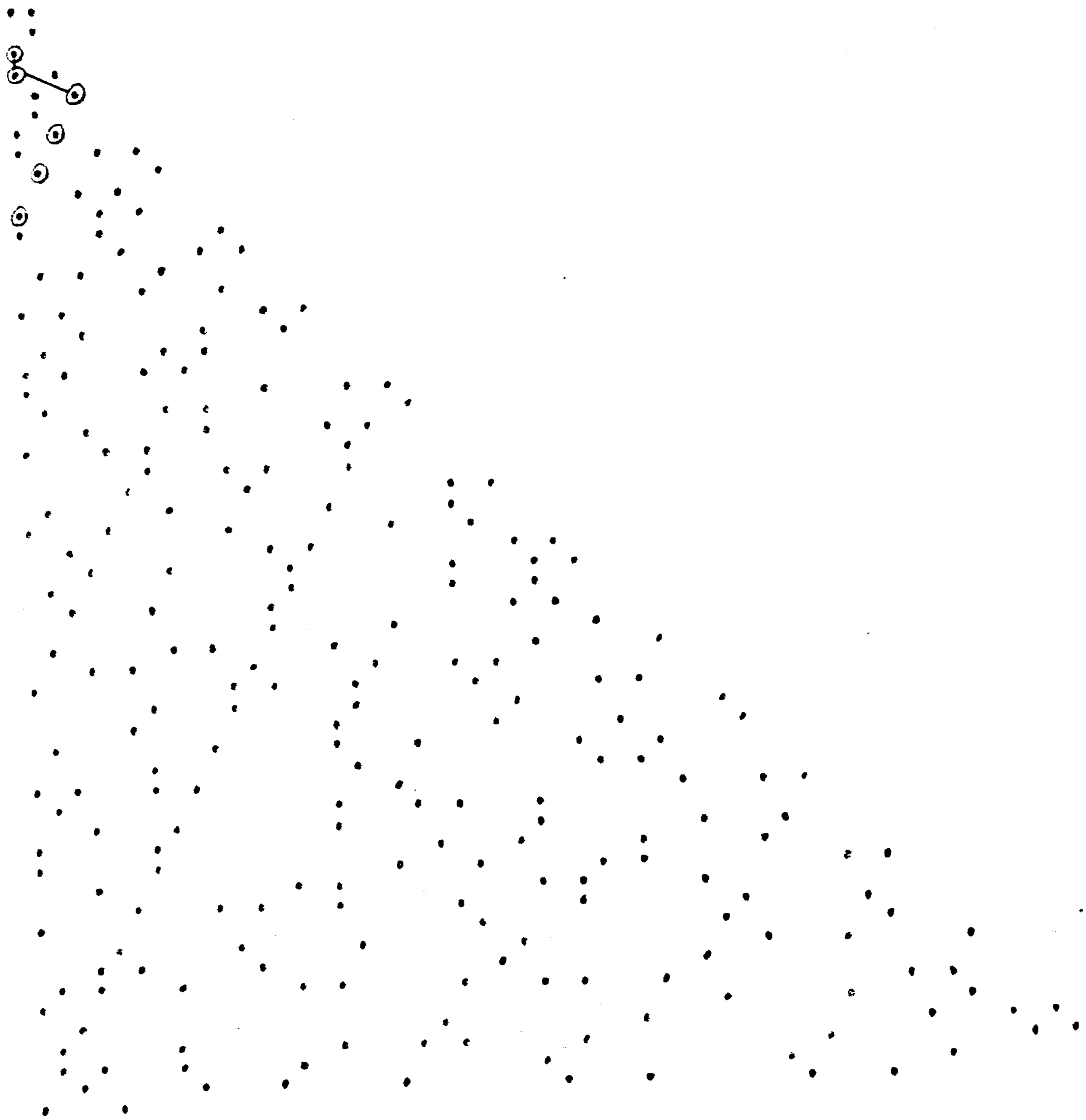

Figure 42.

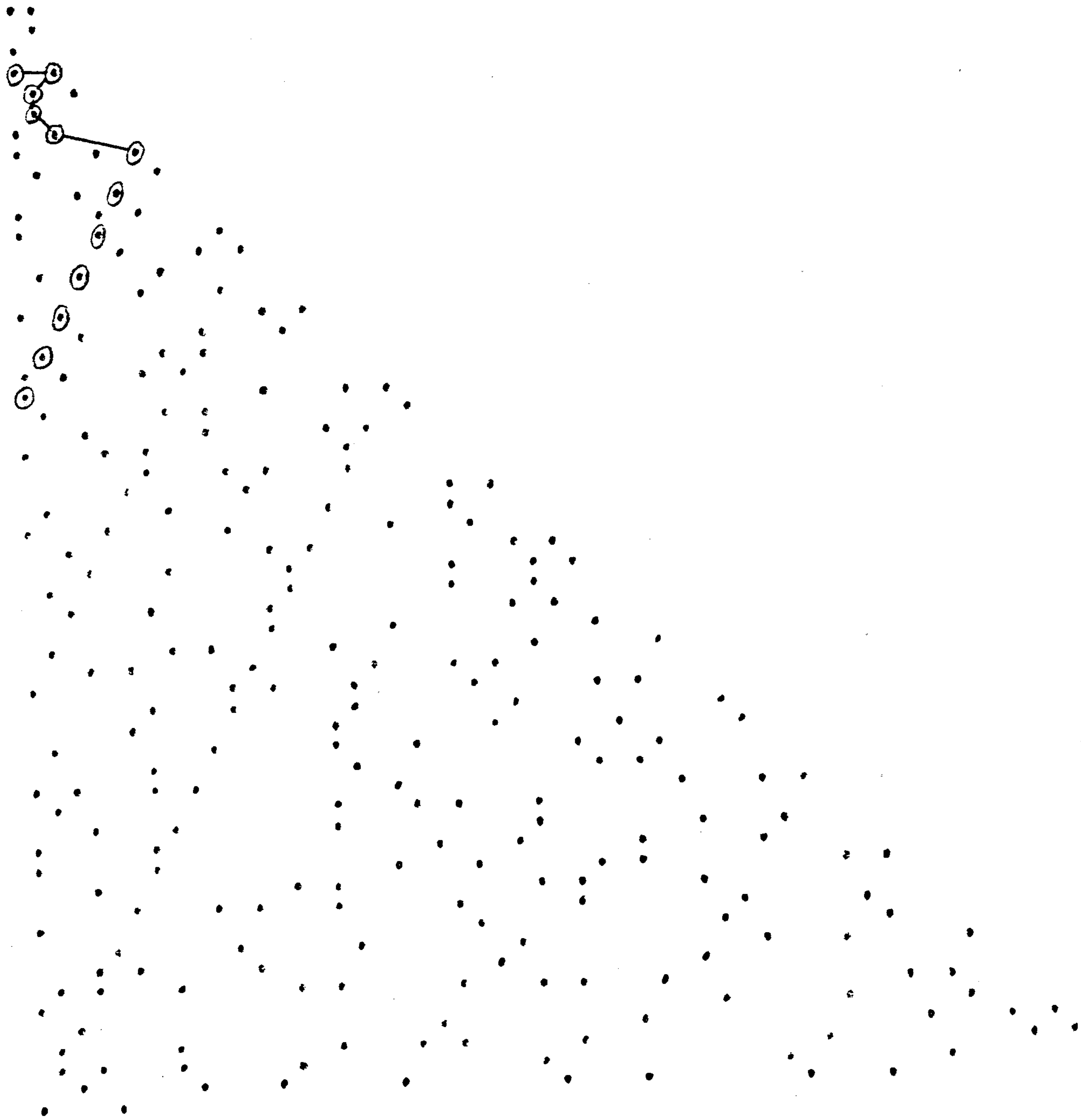

Figure 43.

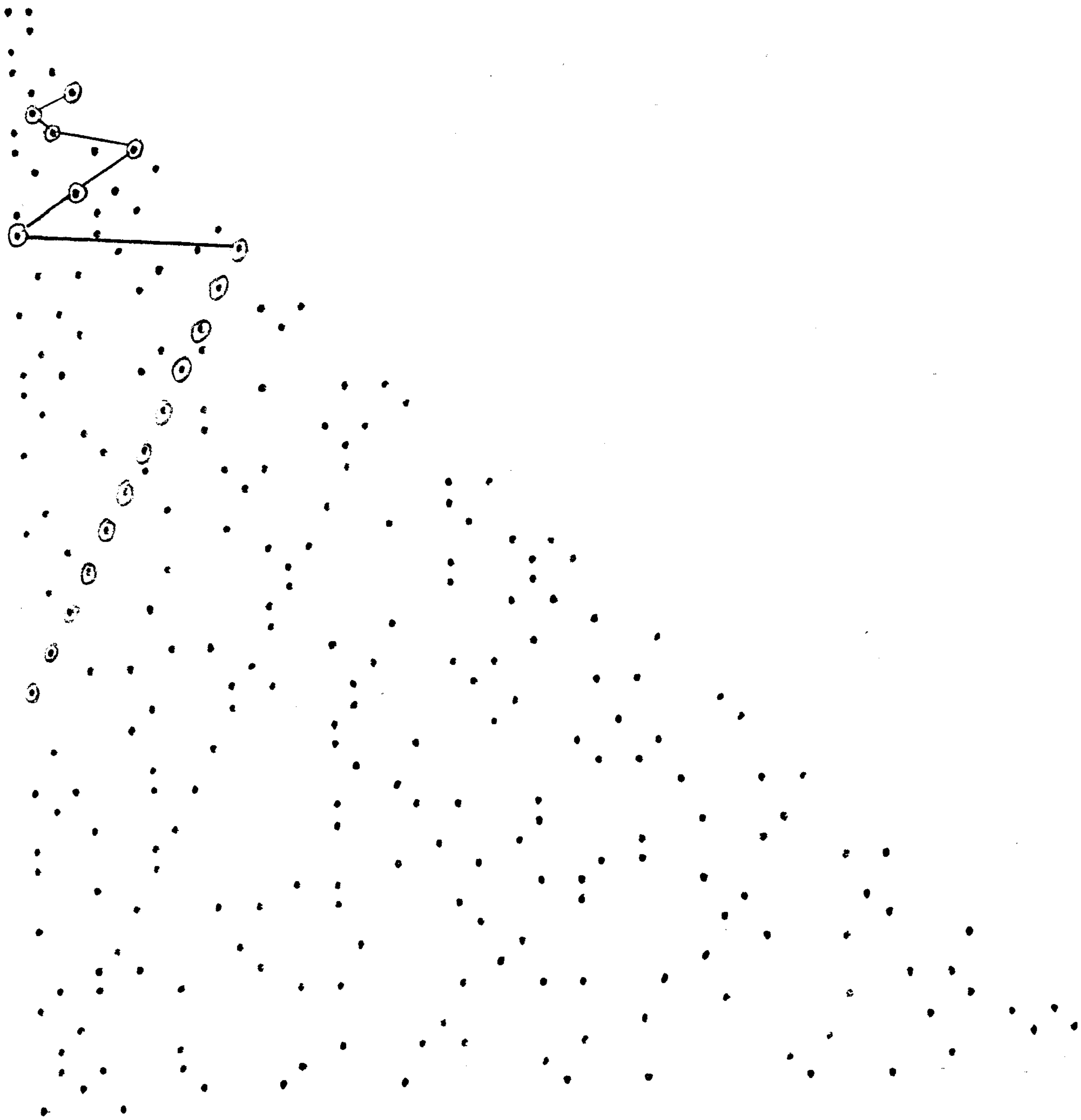

Figure 44.

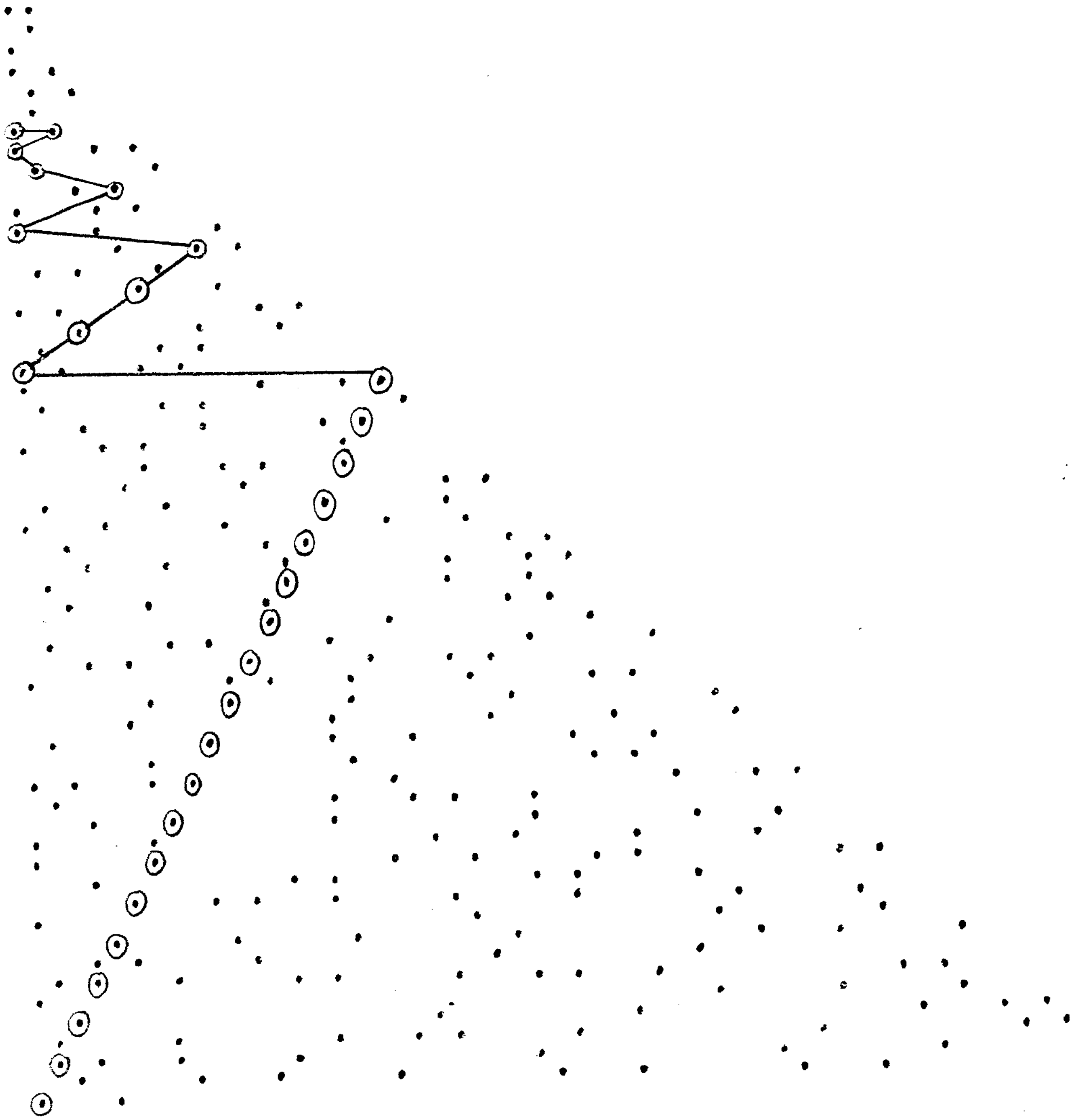

Figure 45.

More recently, though, I wondered what would appear if the unused area "northeast" of the triangular array could somehow be extended into. What meaning, if any, could be attached to the points projected to by continuing the straight lines into that unused portion of the graph? For example

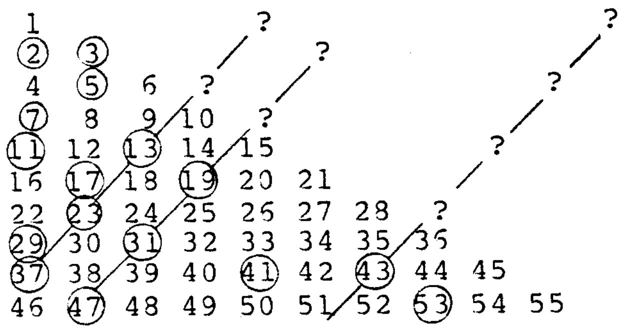

Figure 46.

Well, a natural approach to getting a handle on the spots designated by question marks would be to count over from the last spot on the same line of the triangular array. This would yield the following:

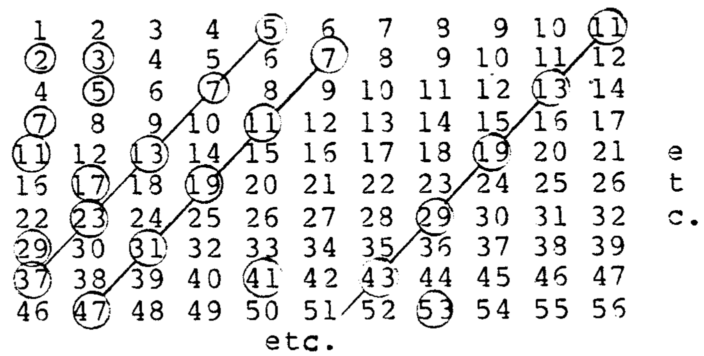

Figure 47.

Voila! The question-marked spots are the lower values of each given $2x^2$ formula - all the lower values! (See figure 48.) What a satisfying result.

When the unused portion of the graph of the triangular array is fully extended into by counting across from the "hypotenuse" and all occurrences of primes are singled out as in figure 49, new patterns of primes, encompassing the whole graph, appear (See figures 50-54.).

Not only are there unbroken straight lines of "primes" but unbroken curves, too (See figures 55-58.).

Are any of these new patterns really new? No. Each

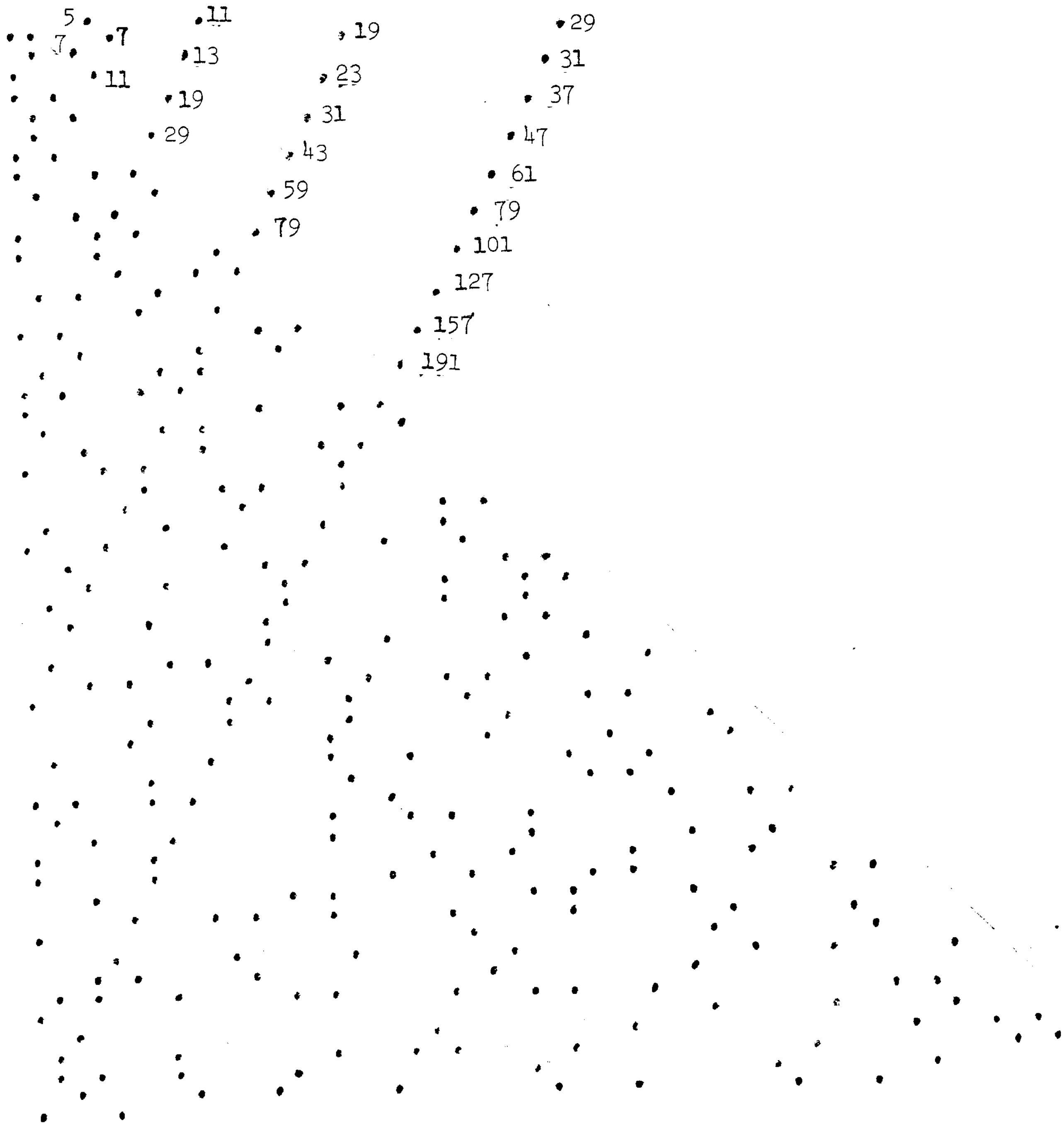

Figure 48.

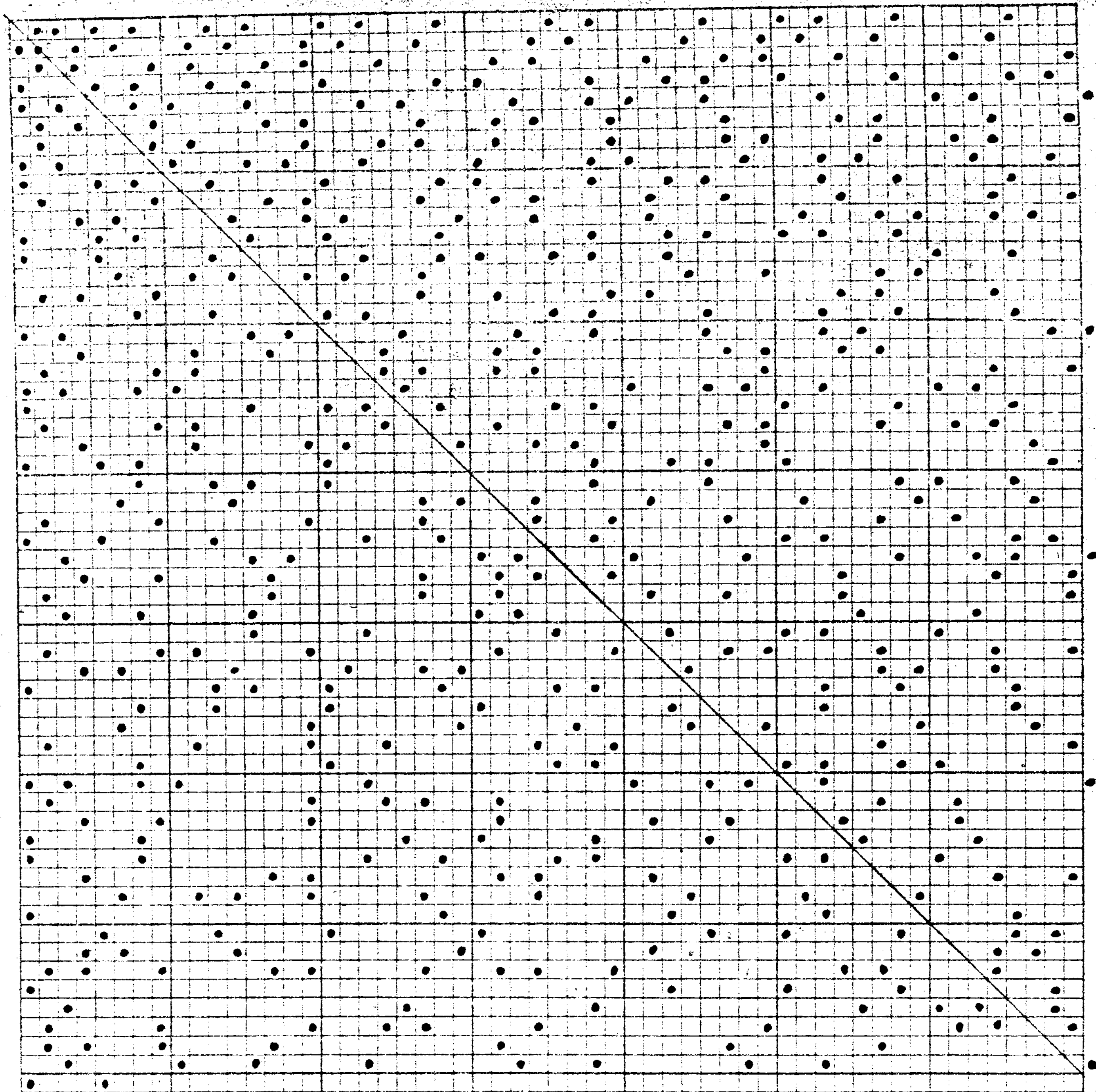

Figure 49.

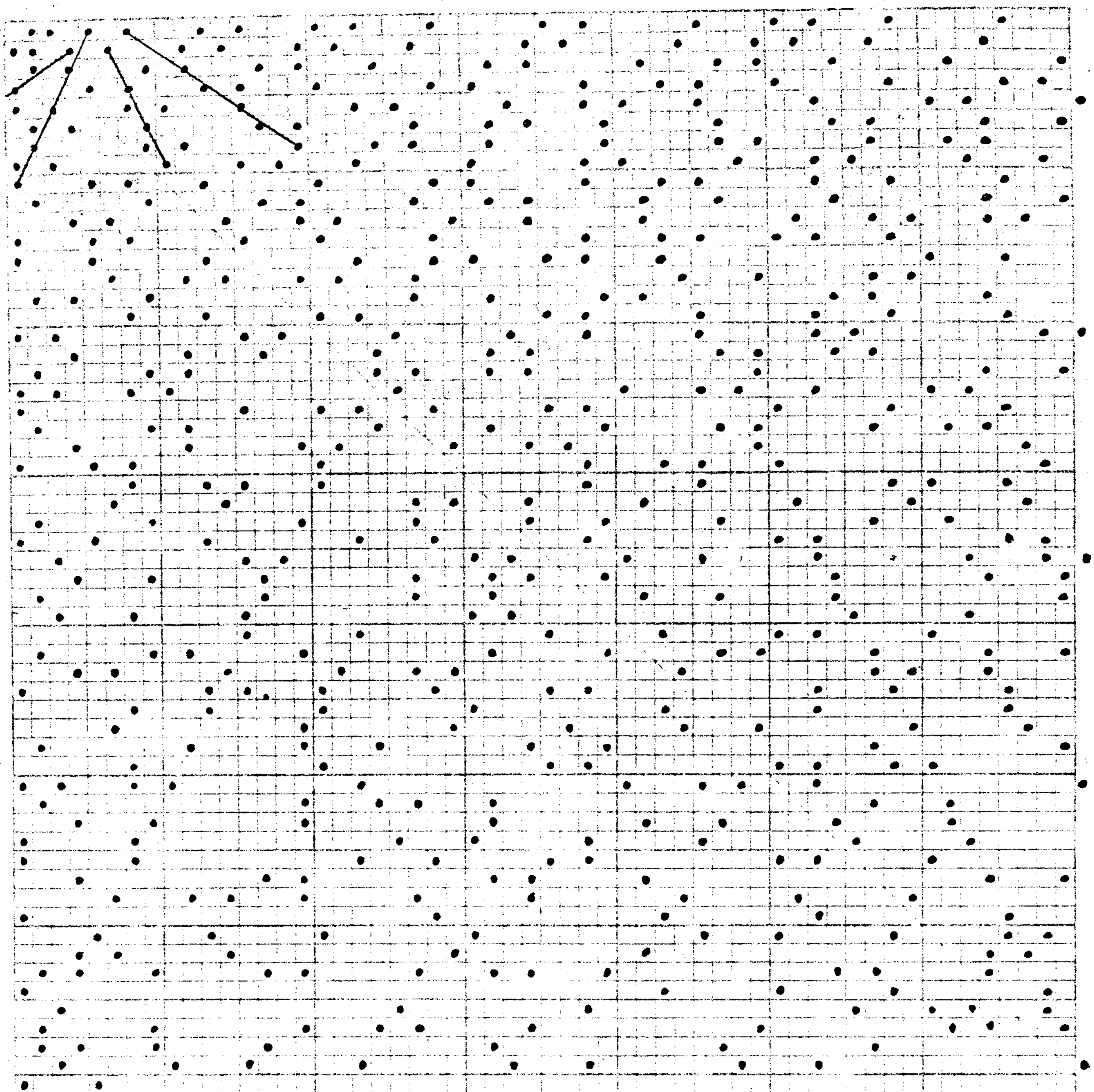

Figure 50.

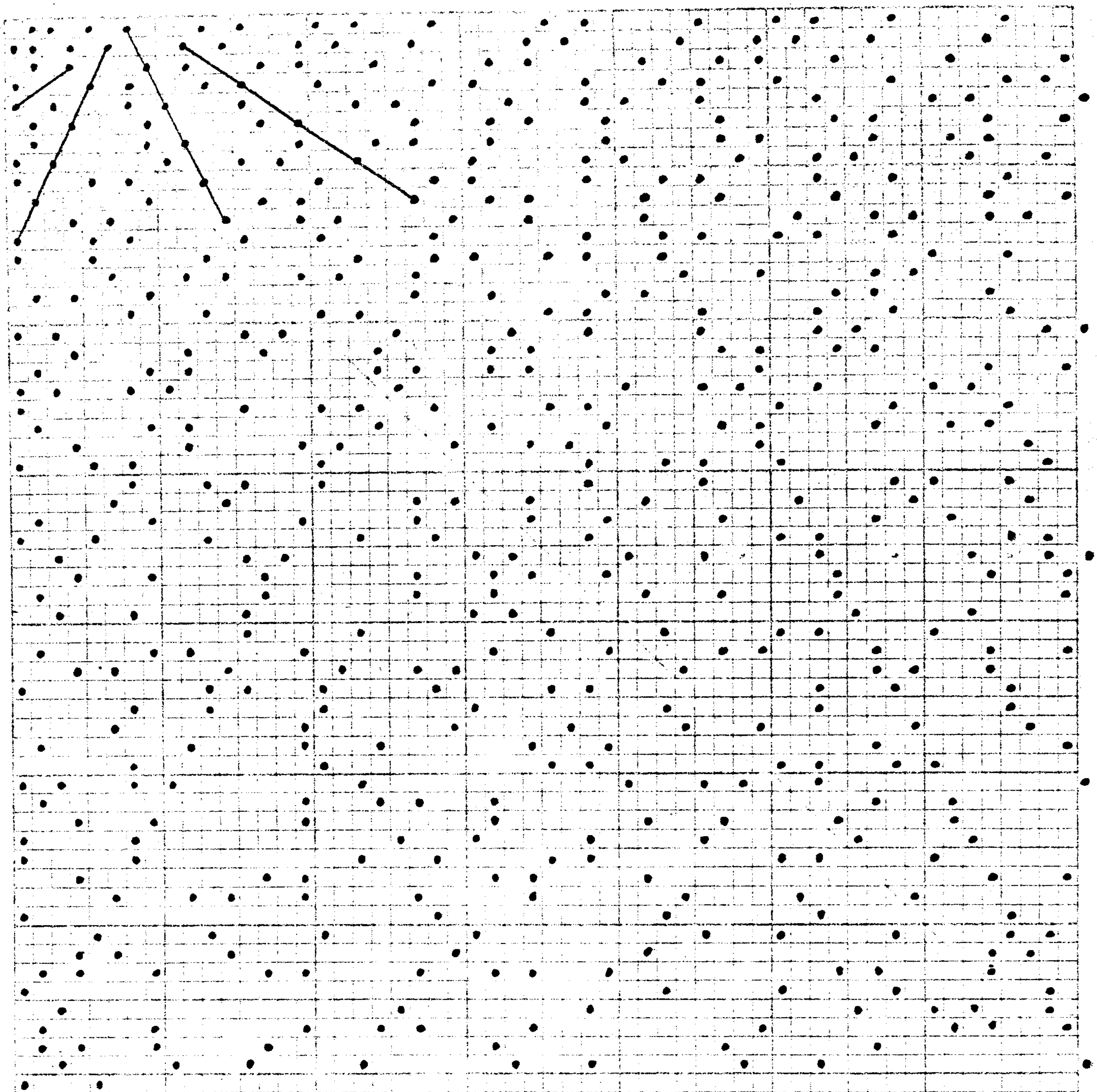

Figure 51.

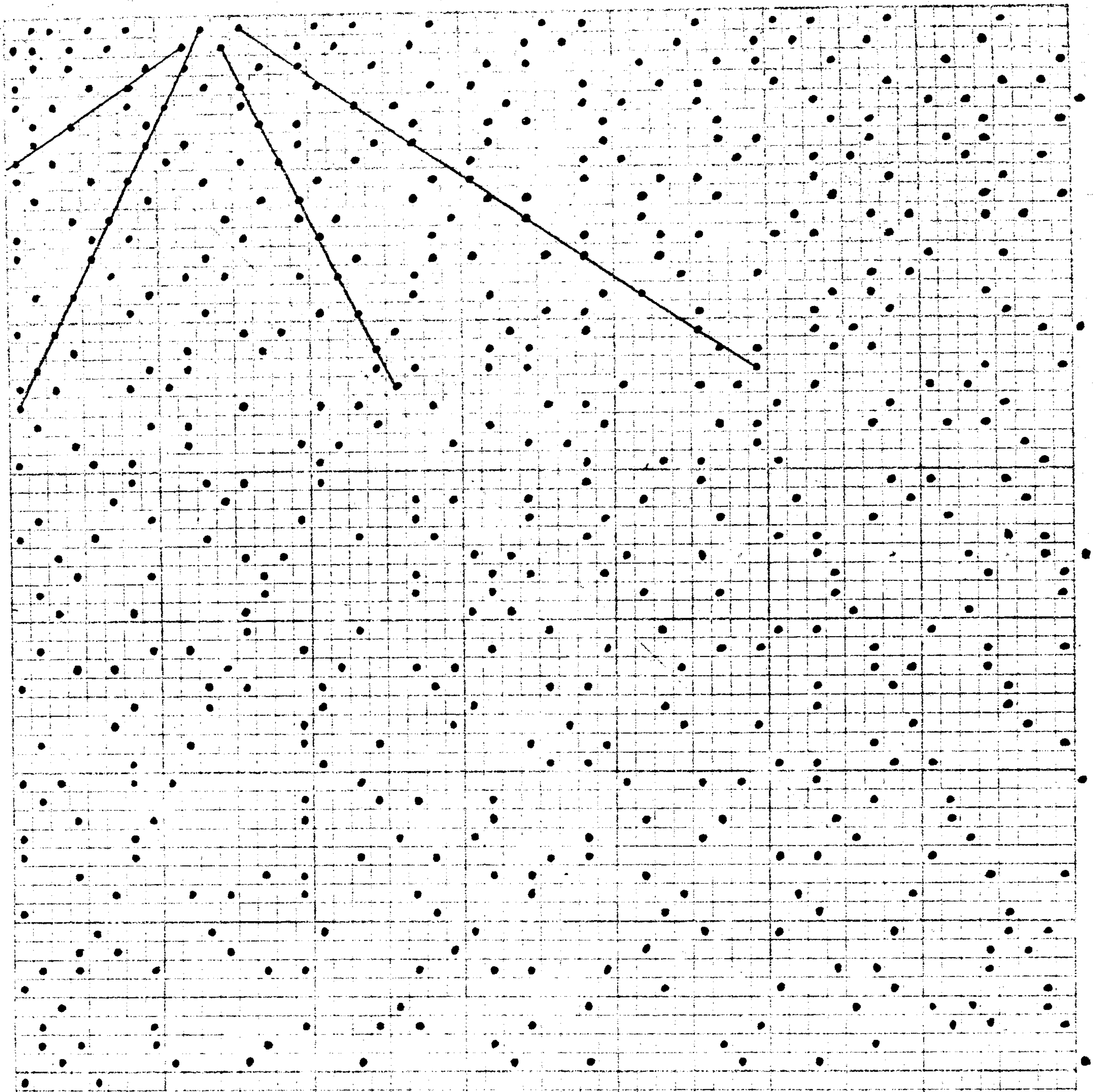

Figure 52.

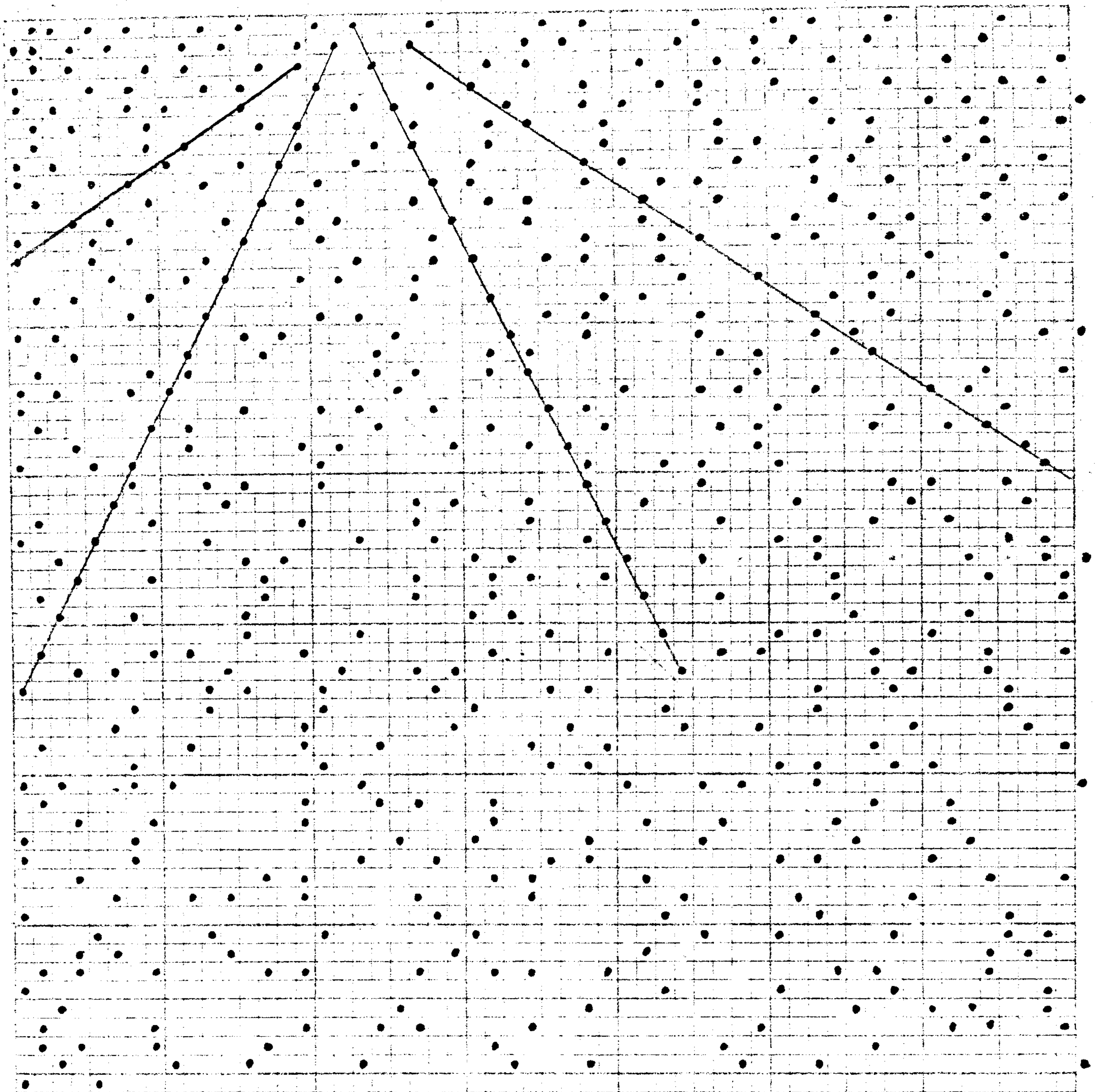

Figure 53.

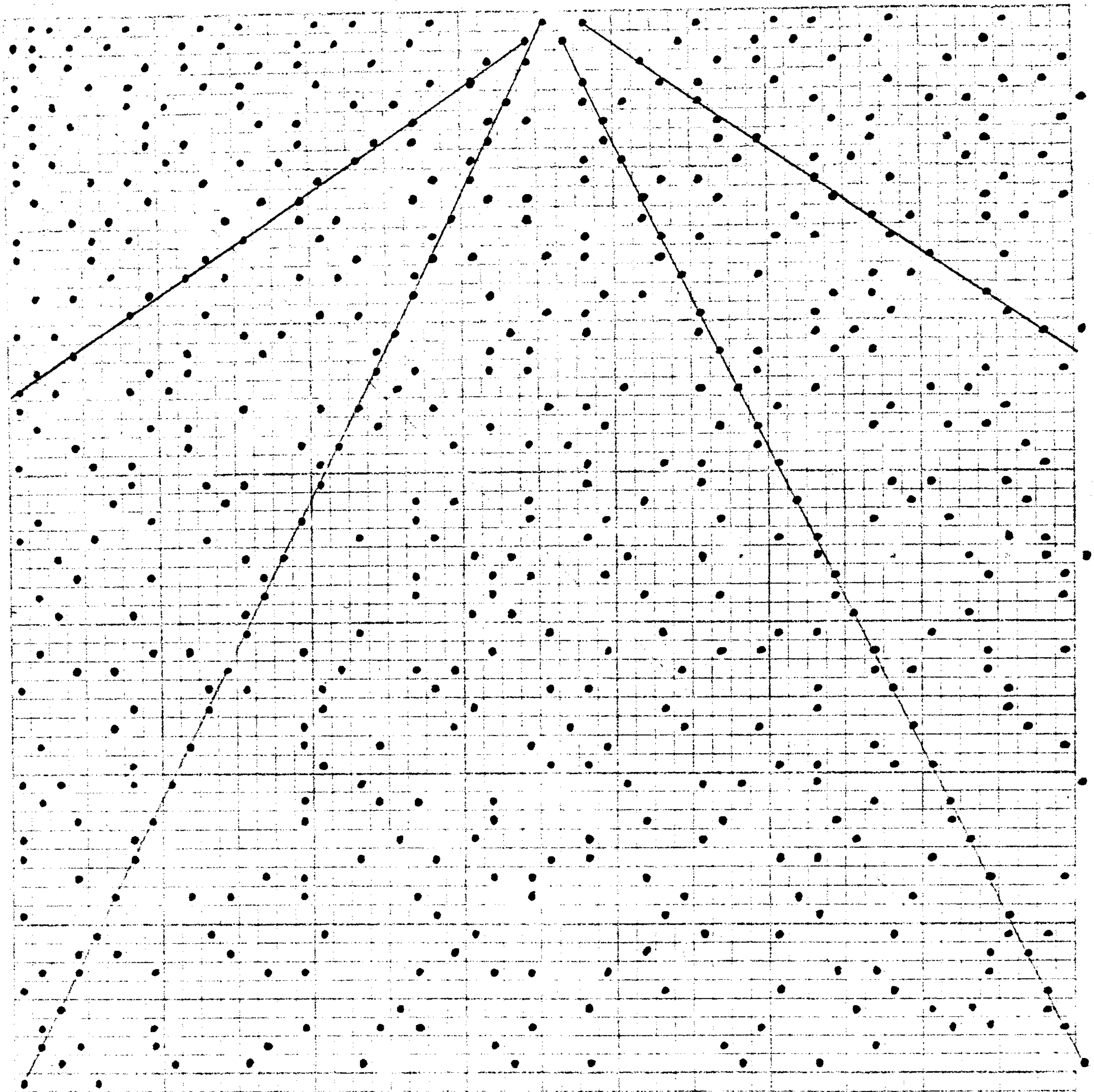

Figure 54.

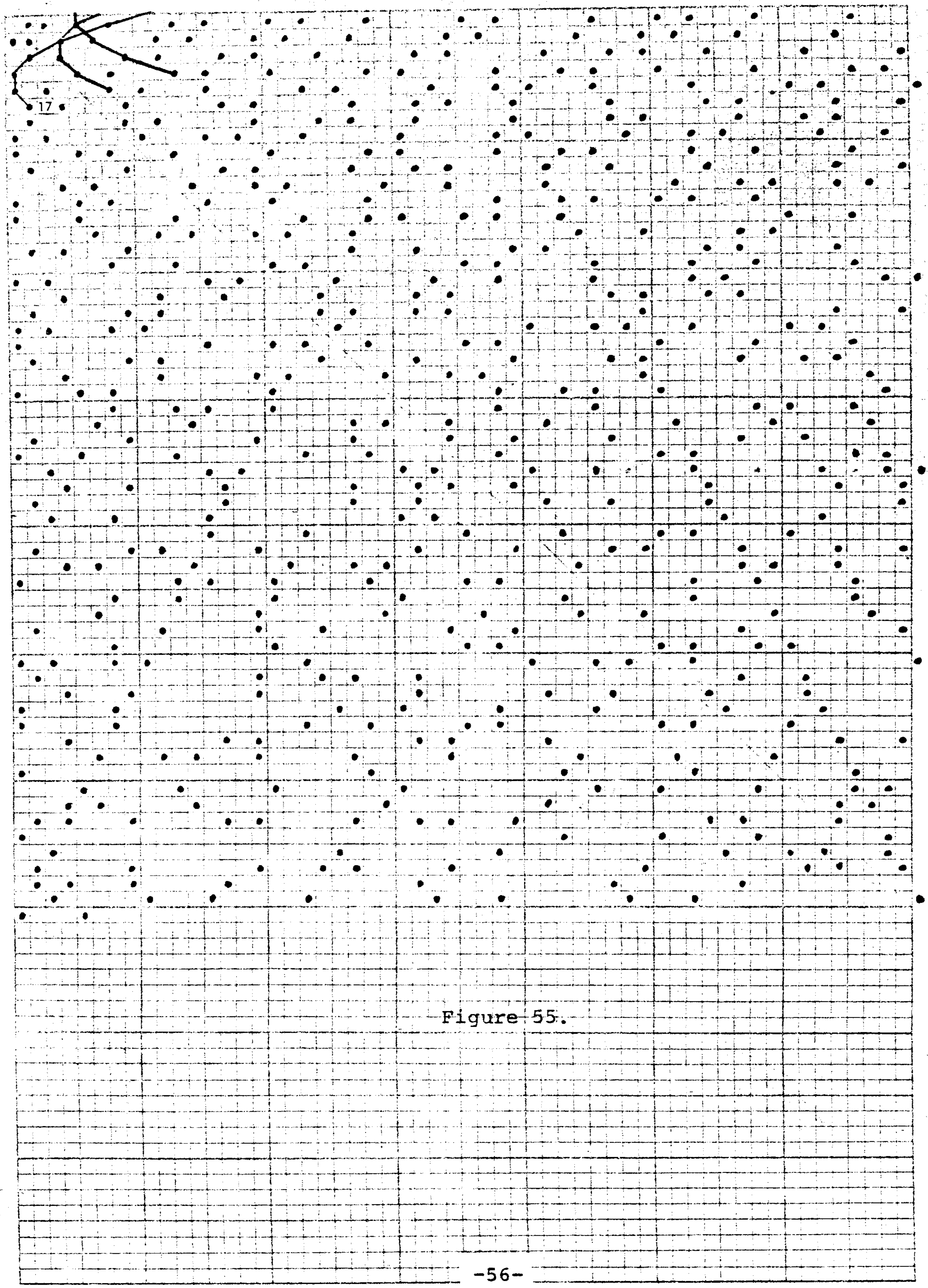

Figure 55.

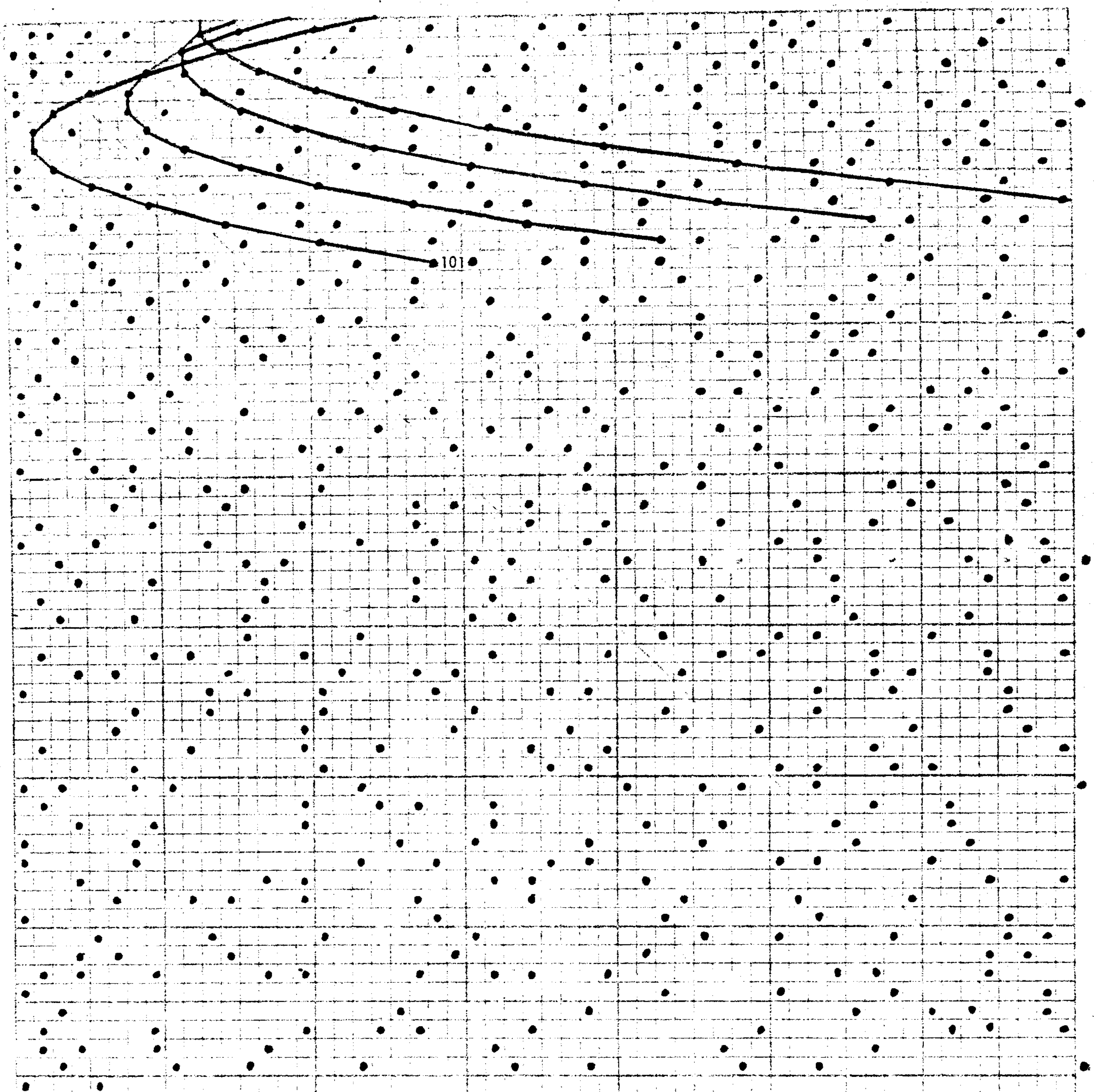

Figure 56.

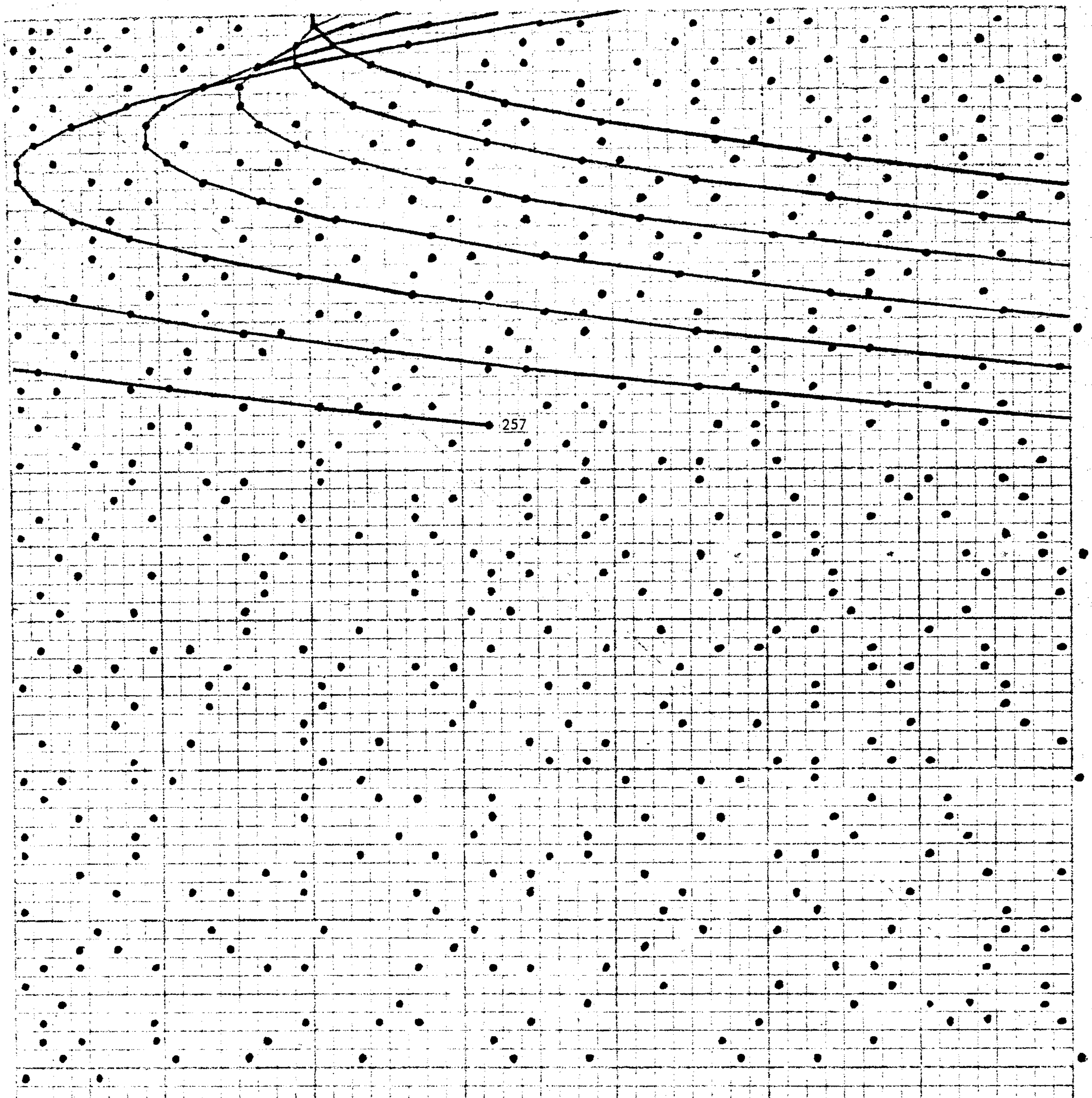

Figure 57.

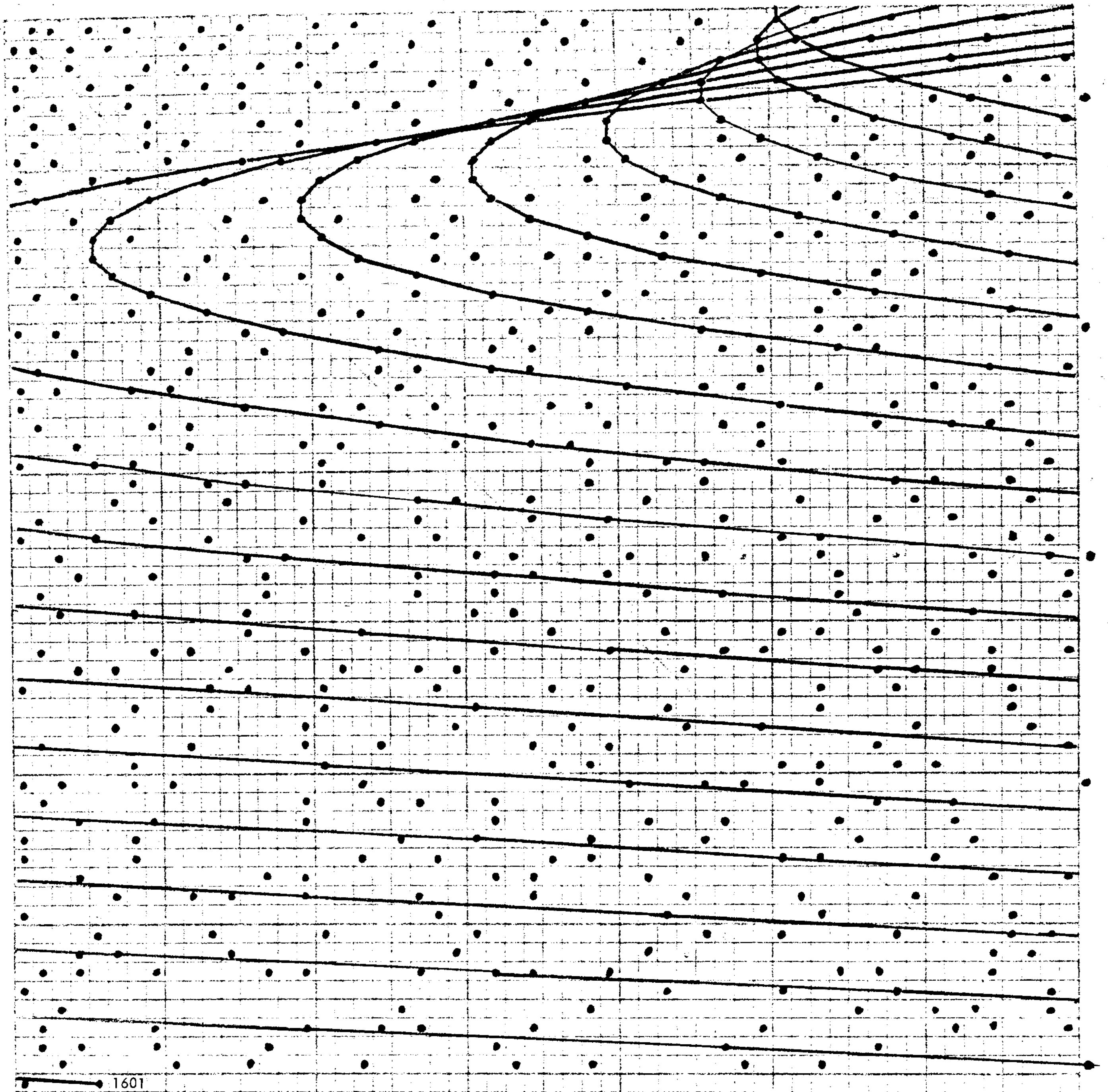

Figure 58.

new straight line is the same series as an original straight line, while each curve family belongs to its respective "design" (each $x^2 + x$ sequence generates "the design"). The difference between the four $x^2 + x$ "designs" is their displacement from the y-axis of the triangular array.

11. UNBROKEN PRIME RIBS

I was recently perusing a graph of the triangular array that I had made ten years ago, the one which had originally led me to extend $x^2 + x + 41$ outside the triangular array. This graph contains each $x^2 + x + c$ sequence easily identified by its own color ink, each represented as far as it gives primes for consecutive values of x.

Looking at the "ribs" of $x^2 + x + 17$, I decided to see if any primes would be hit if the "ribs" were extended beyond their existing limits, down the triangular array (See figure 59.).

Extending the rightmost rib down, the next spot it would hit is spot 323 (coordinates line 25, spot 23). 323 is not prime (323 = 17 x 19). Extending the rib any further takes it outside the triangular array.

Extending the center rib down, we first stop at 289, not a prime (289 = 17 x 17), then at a prime, 397. From there the rib hits another prime, 523. After that the rib leaves the triangular array.

Extending the left rib of $x^2 + x + 17$ yields a surprise--all the spots extended to are primes: 359, 479, 617, and 773 (See figure 60.).

Wondering whether any other such prime ribs might lie waiting to be uncovered, I began testing some $x^2 + x + 41$ downward ribs. I tested the five ribs indicated in figure 61 from left to right.

The first, third, and fifth ribs hit a non-prime right away. The second rib was a real tease, though. For the first eight spots extended to, a prime is the resident: 1847, 2297, 2797, 3347, 3947, 4597, 5297, and 6047. The next spot is not a prime, however: 6847 = 41 x 167.

The fourth rib hits only primes: 1933, 2393, 2903, 3463, 4073, 4733, 5443, 6203, 7013, and 7373.

Counting the four spots prior to 1933 in the fourth rib, 593, 853, 1163, and 1523 (not tested since we already know they're prime, being in $x^2 + x + 41$ where $-1 < x < 40$), the rib has fourteen spots, all of which are prime. (See figure 62.)

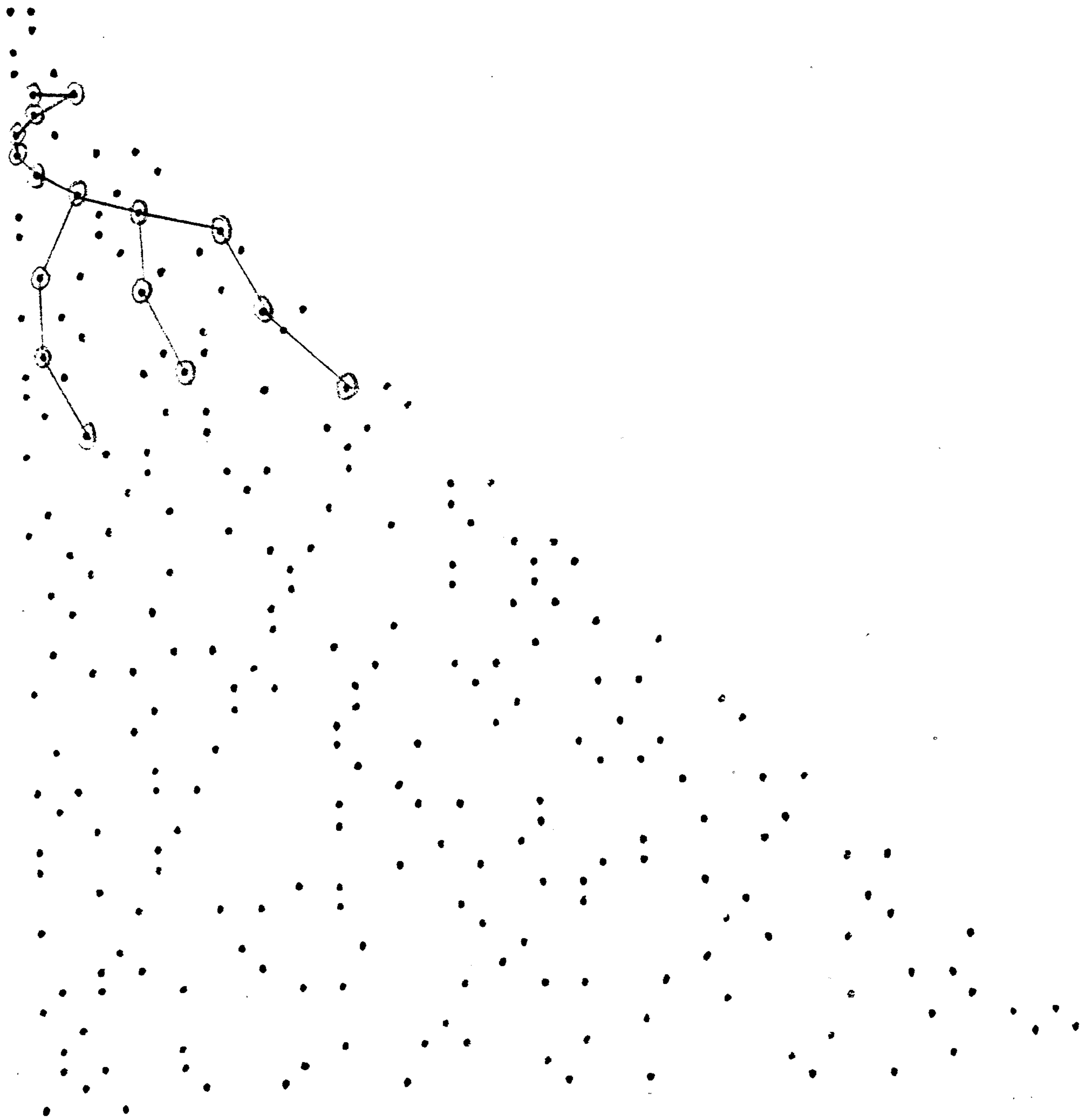

Figure 59.

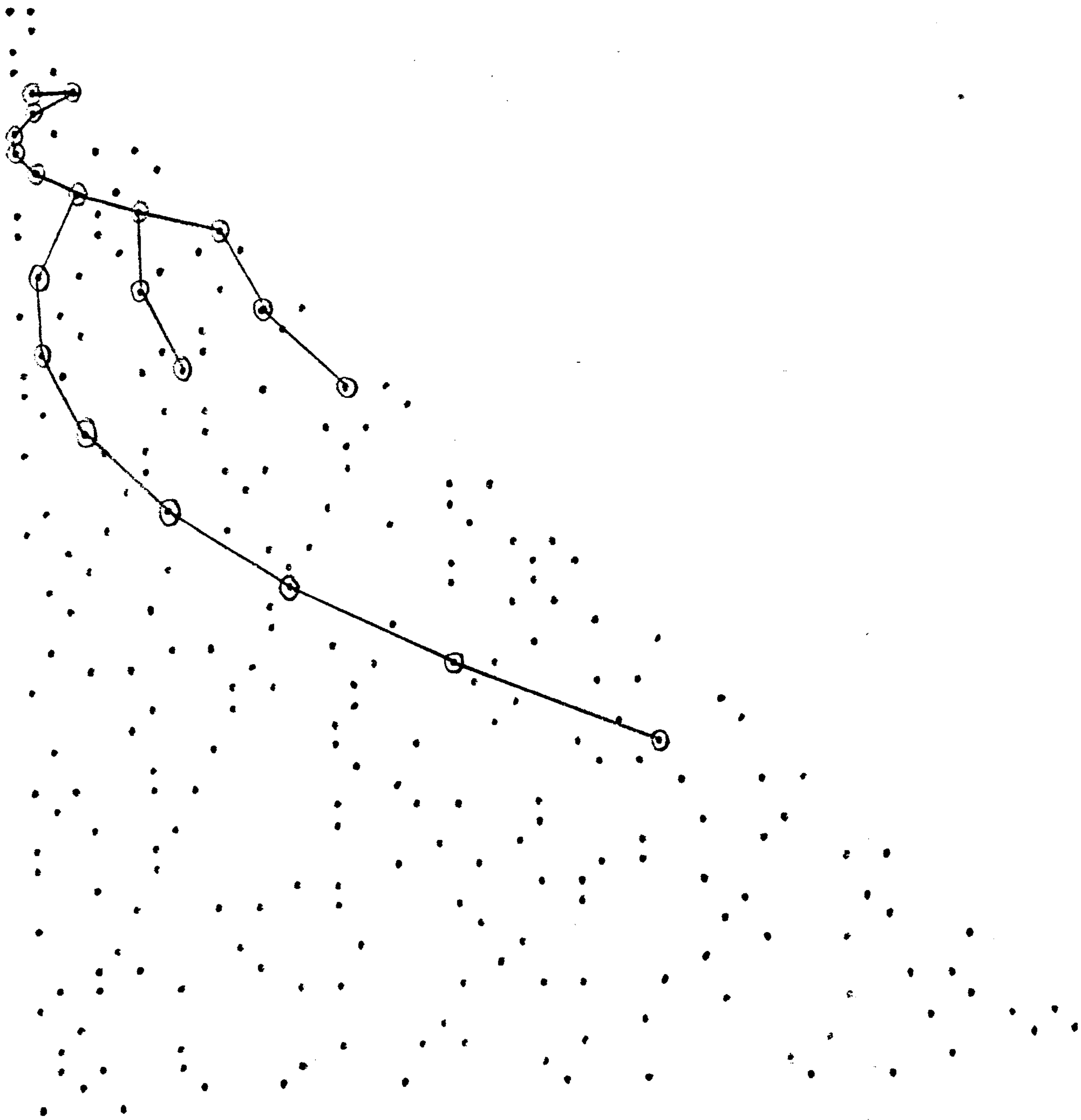

Figure 60.

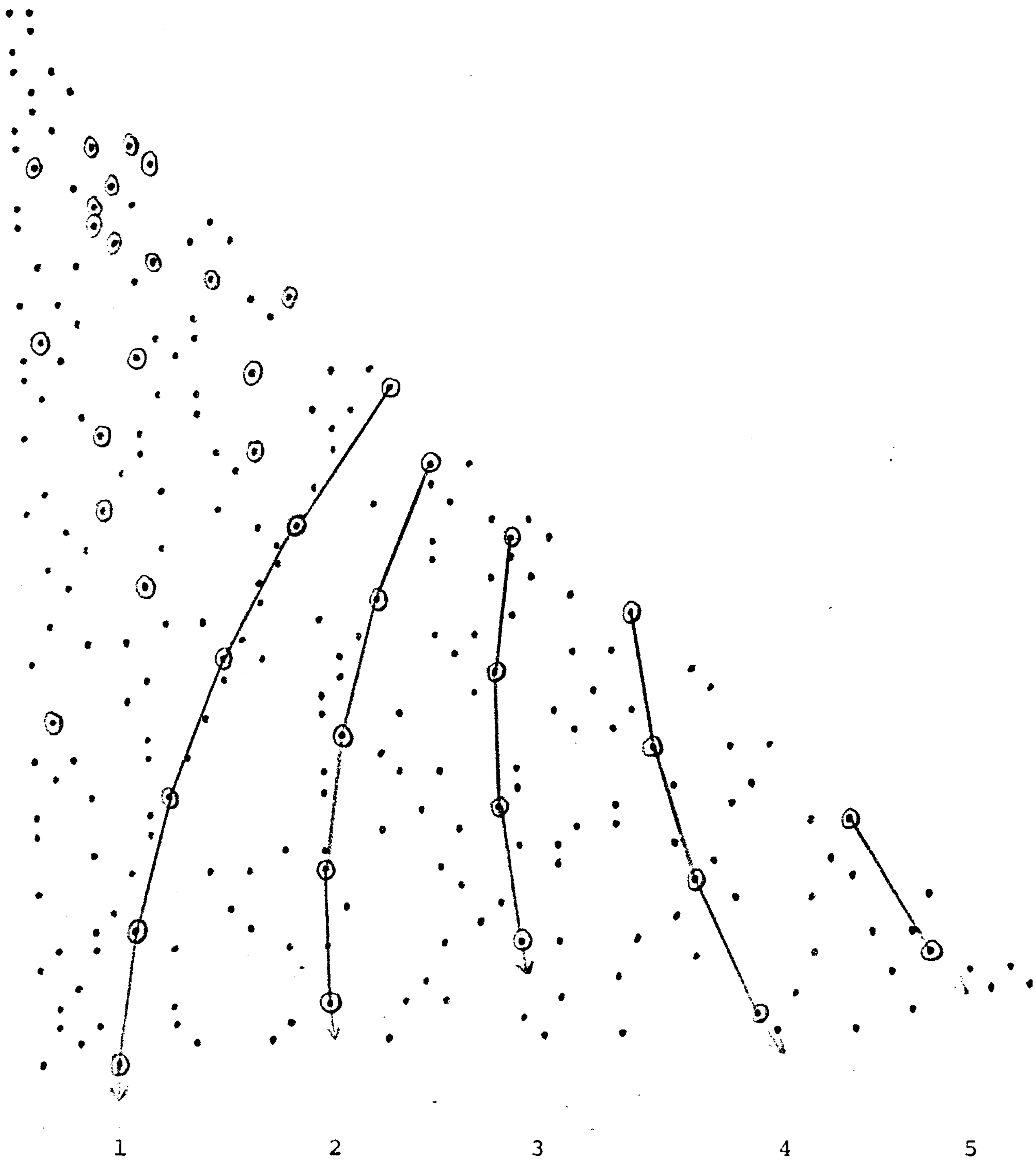

Figure 61.

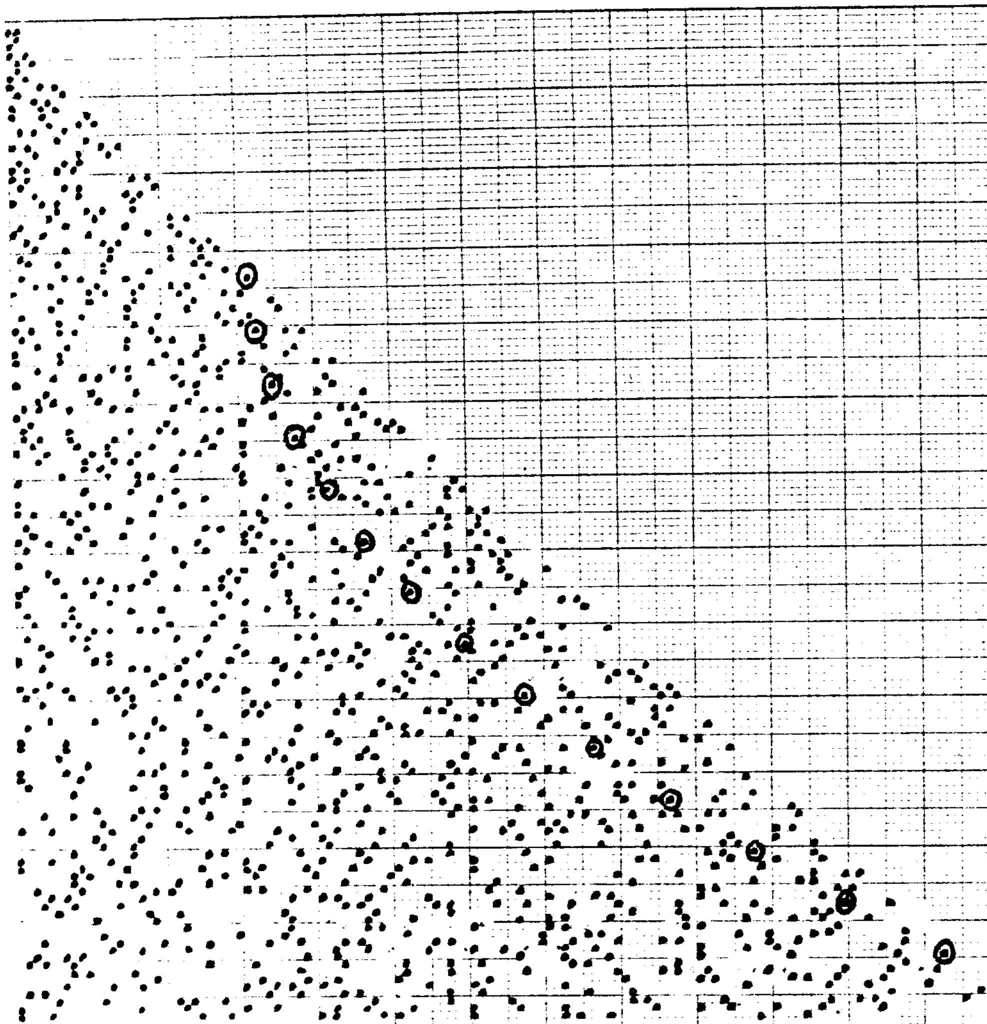

Figure 62.

10 Millimeters to the Centimeter

Perhaps there are more all-prime ribs further down the array - who knows?

12. THE OTHER ARRAY AND THE OTHER FOUR LINES

Another more recent tack taken has been experimentation with variations on the triangular array. For example, what happens to the configuration of primes when we count

```
1 2
3 4 5 6
7 8 9 10 11 12
      etc.
```

Figure 63.

Are there new patterns, like straight lines, to be found? Are there perhaps new sequences and thus formulas lying hidden there, waiting for us to find the right key to unlock these secrets? The answers are, respectively, yes and no.

There are new patterns - in fact, four parallel straight lines of primes - to be seen. The sequences and their associated formulas are not new, however. Indeed, they are our old friends, the four x^2 sequences (See figure 64.).

Trying yet another counting variation,

```
1
2 3 4
5 6 7 8 9
  etc.
```

Figure 65.

the same lines appear, but slanted. This time they look just like the lines of the original triangular array (See figure 66.).

When these four lines are extended into the unused "northeast" area of the graph, the spots projected to also correspond to all the lower values in each given sequence (See figure 67.).

13. $2X^2 + 2X$ ON THE OTHER ARRAY

I decided to see what design would appear, if any, like the curving one made by $x^2 + x$ on the original triangular array, when I plotted $2x^2 + 2x$ on the new triangular array and then extended the curves beyond the new array. The new

Figure 64.

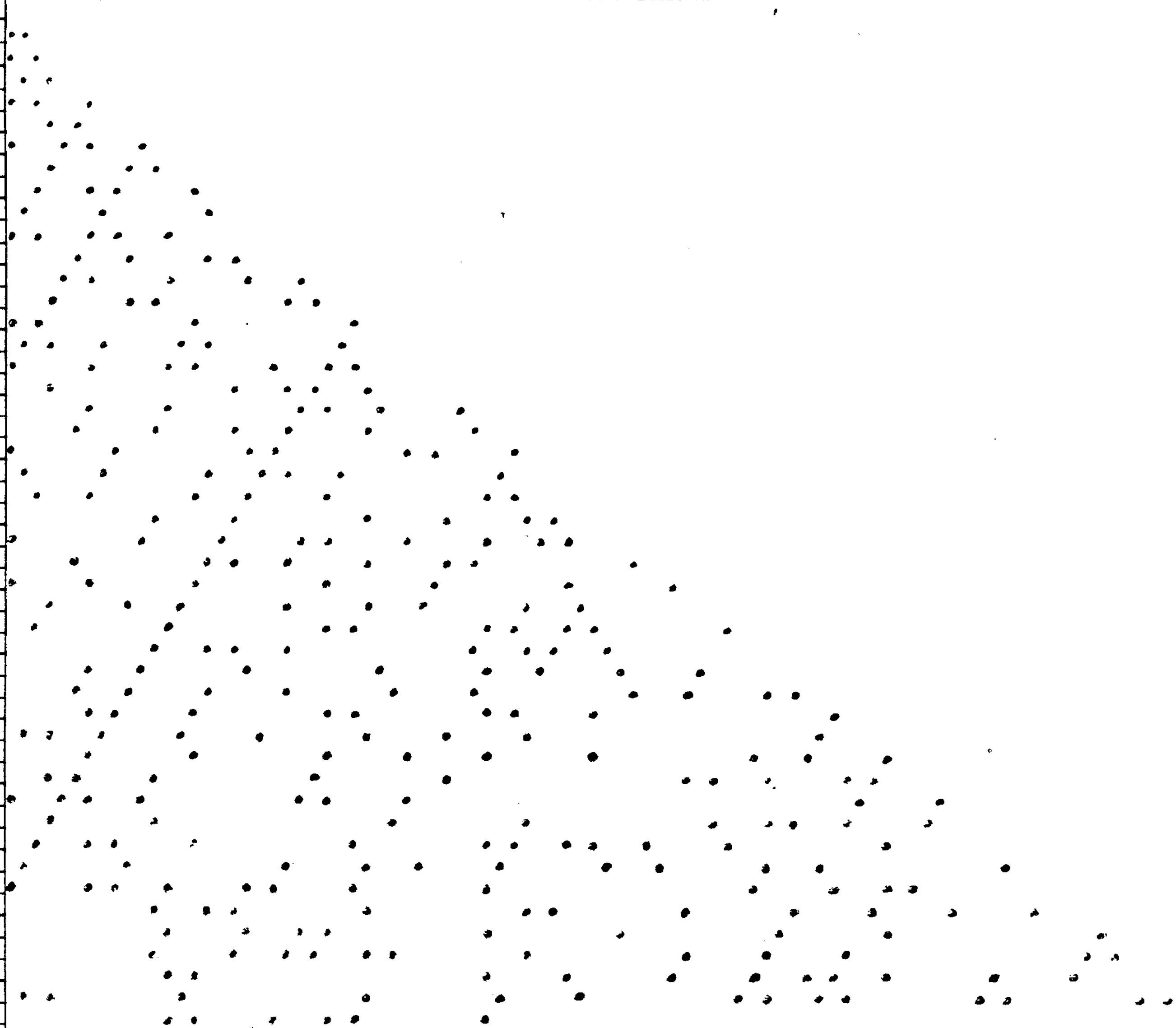

Figure 66.

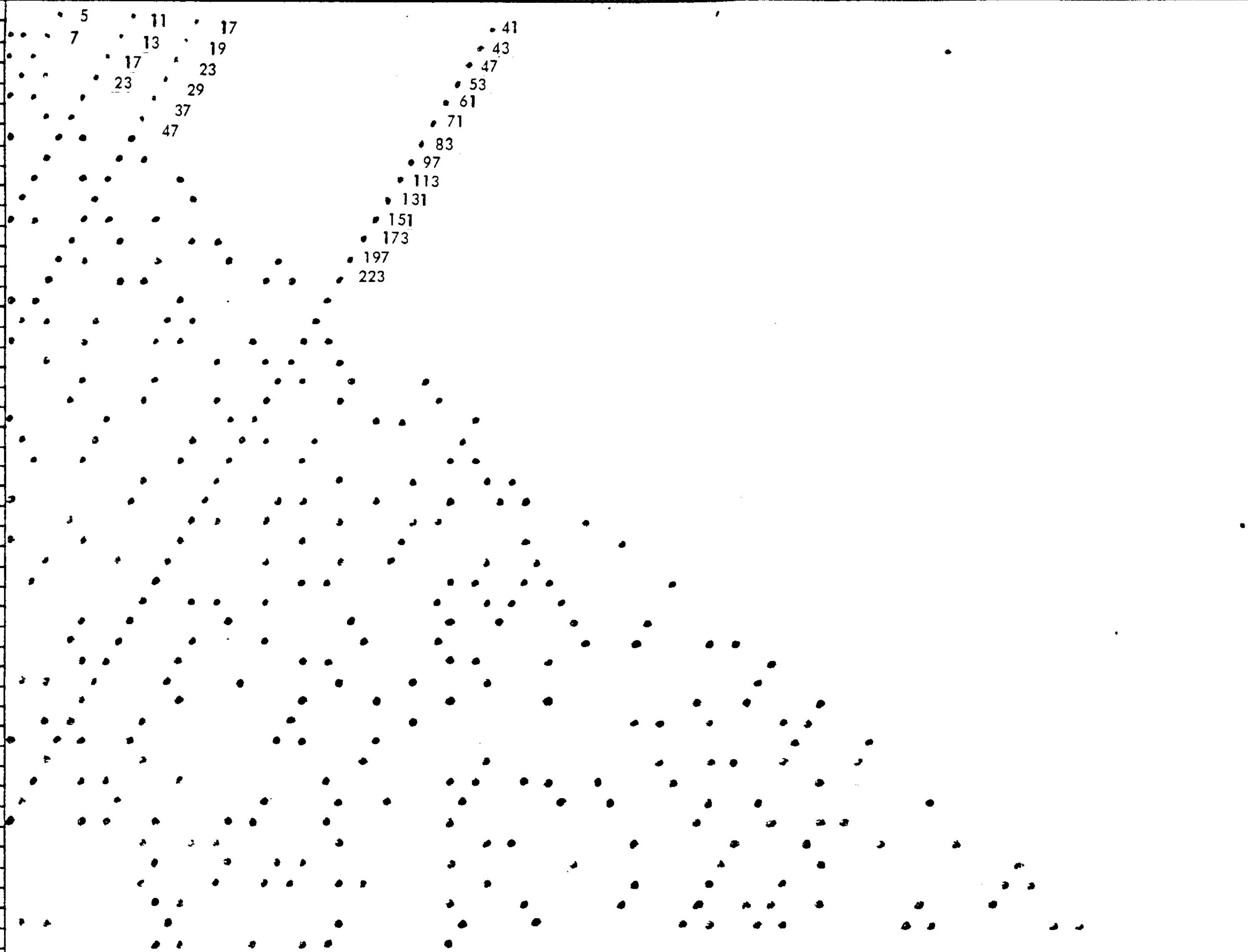
5
7
11
13
17
23
17
19
23
29
37
47
41
43
47
53
61
71
83
97
113
131
151
173
197
223

Figure 67.

"design" turned out to be very similar to the original one, the only difference being the format and spacing of curve family members (See figure 68.).

14. "DESIGN" - MULTABLE PARALLELS

There is an interesting resemblance between two unrelated curve families: the right void adjacent curve family of the new "design" (when viewed with the void at the bottom) and a nested family of curves in a multiplication table that I have devised. The multiplication table is shown in figure 69.

In this table the x-values are products. The factors are a) the y-value and b) the number of dots over from the y-axis. For example, moving horizontally to the right from the y-axis at height = 11, we first encounter a dot at 11 on the x-axis, the second dot across is at 22 on the x-axis, and so on.

By arranging the multiplication table in this manner, prime numbers appear as empty vertical columns between the x-axis and the top diagonal on the graph. (See figure 70.)

When this graph is viewed from the right edge, looking toward the apex at the origin, there appear nested curves of dots as in figure 71.

One is first struck by the fact that all the dots in the table belong to one curve or another in the family (See figure 72.). So all composite (non-prime) numbers fall in these curves. It turns out that every other curve in the family, each curve with two dots at its vertex rather than only one, consists solely of even numbers (remembering that each dot represents a product). That being the case, only the one-dot-vertex curves (again, every other curve) genuinely eliminate odd candidates for primality, the candidates more difficult to crack at a glance.

These odd-depositing curves are of the format depicted in figure 73. The vertex is at a perfect square. The next two dots are over one and up one, the next two dots are over one and up three, the next two are over one and up five, and so on.

The spacing between family members is the distance between successive curve vertexes: 1, 3, 5, 7, ... (See figure 74.).

The new void adjacent families have members with a single dot vertex as well. In fact, these curves are of the same exact form as those of the multiplication table: vertex, two dots one over and one up, two dots one more over and three up, two dots one more over and five up, and so on.

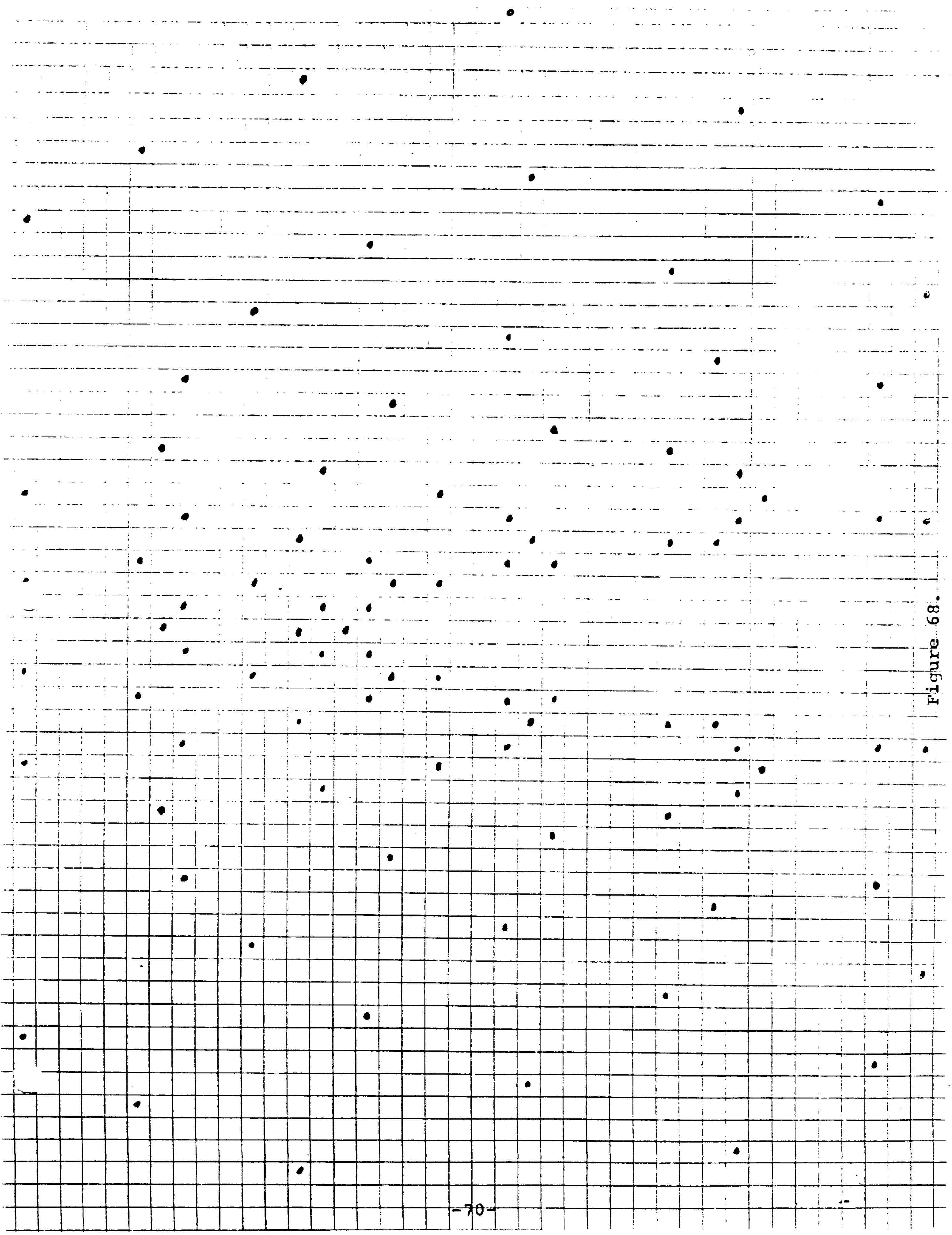

Figure 68.

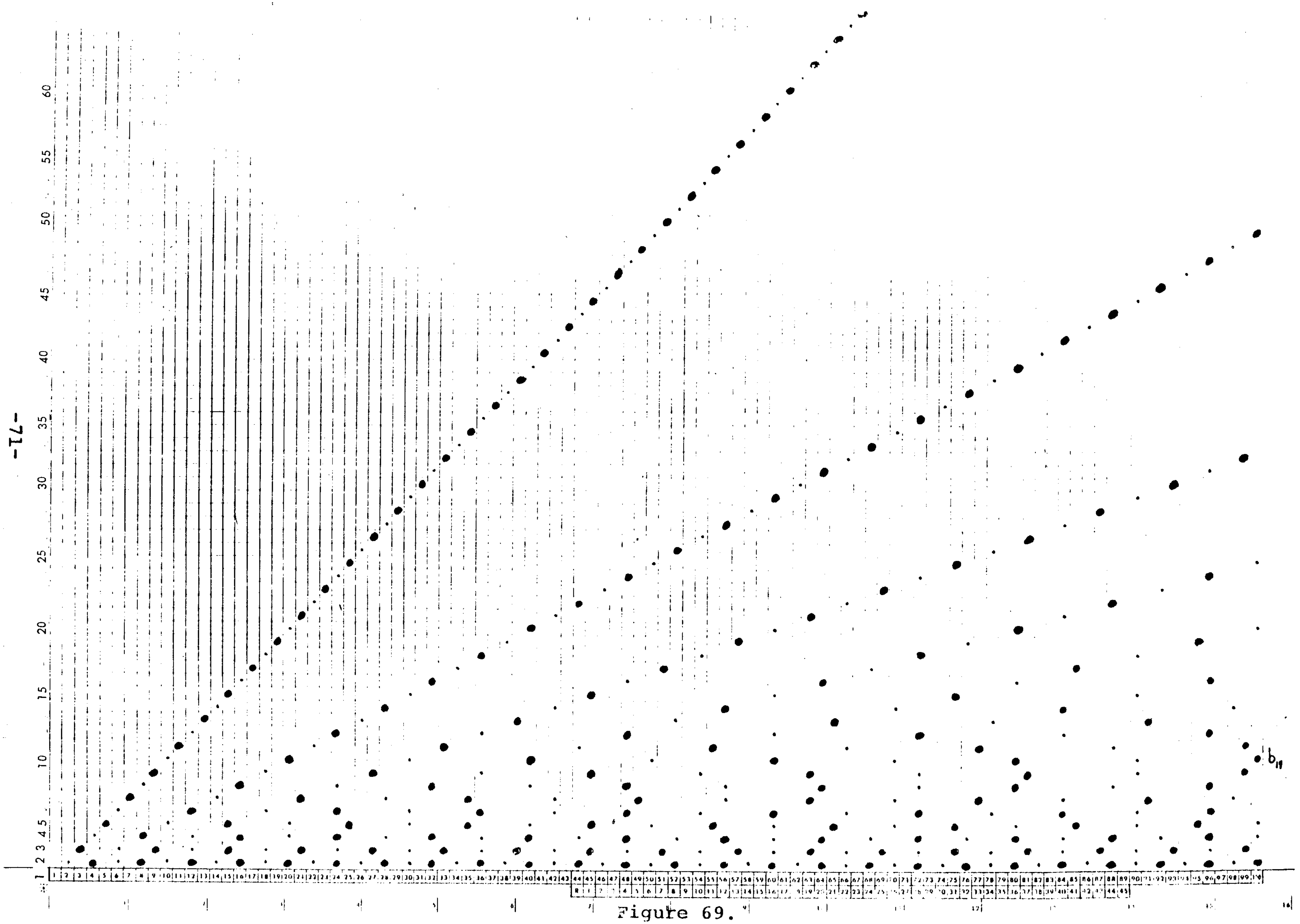

Figure 69.

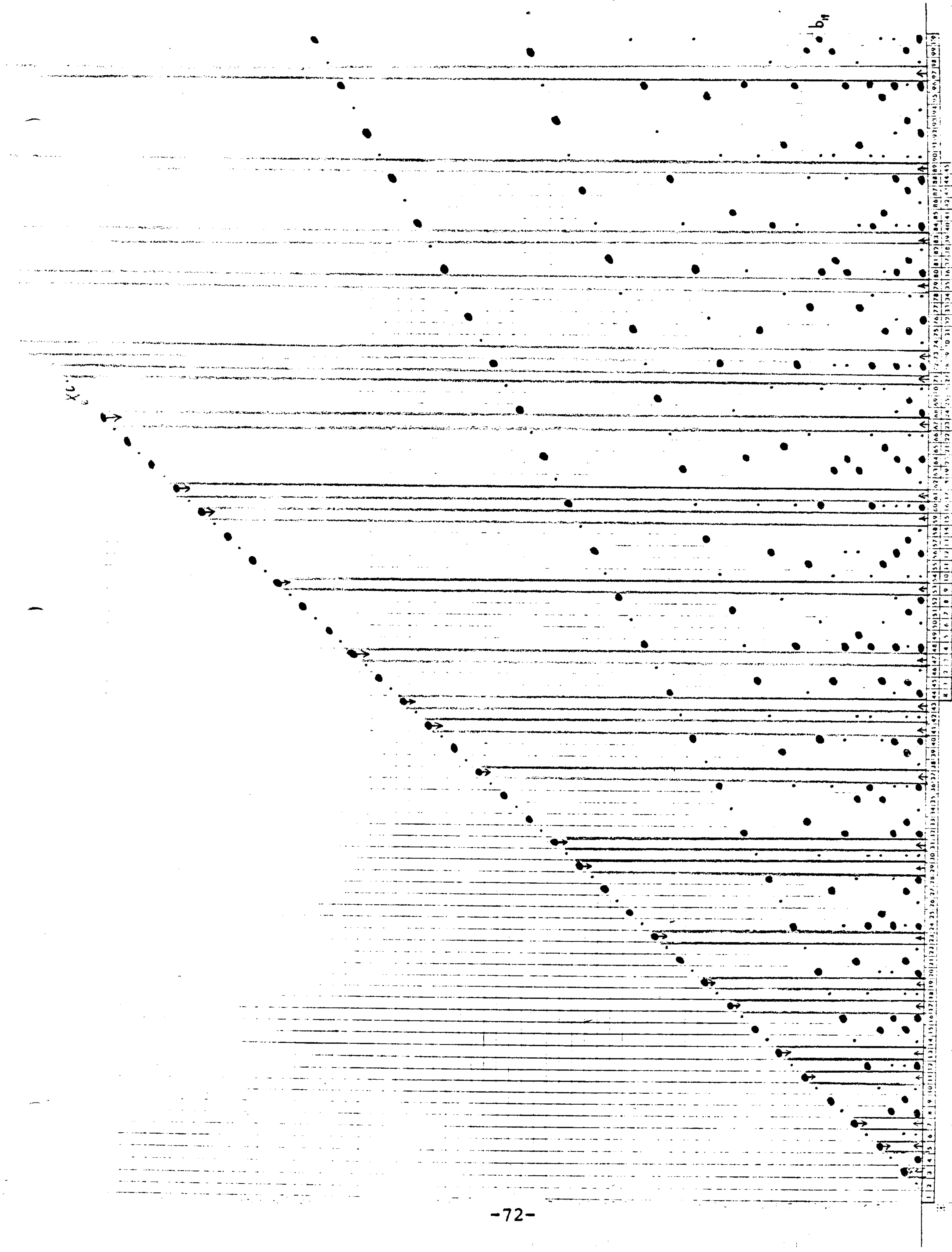

Figure 70.

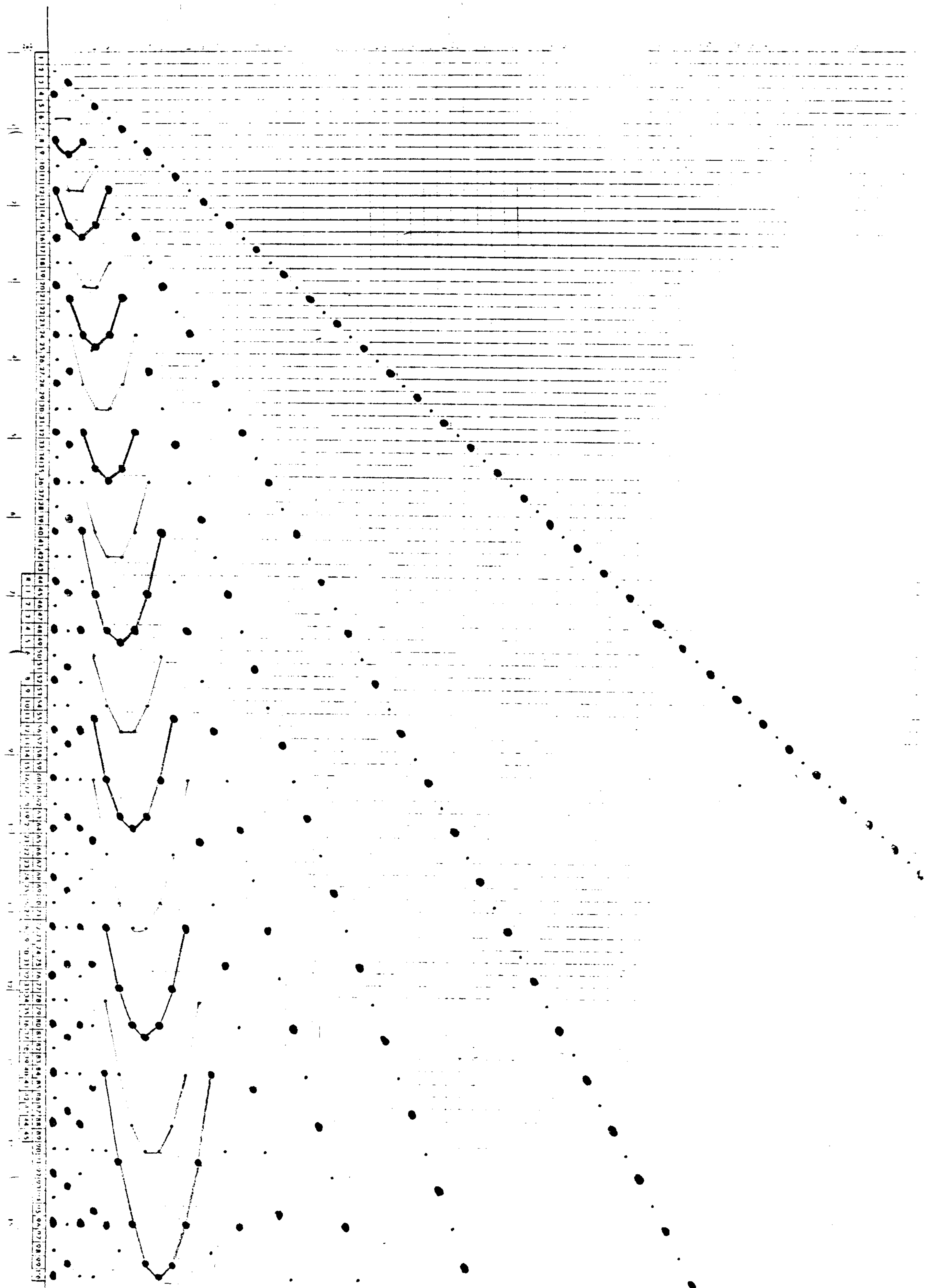

Figure 71.

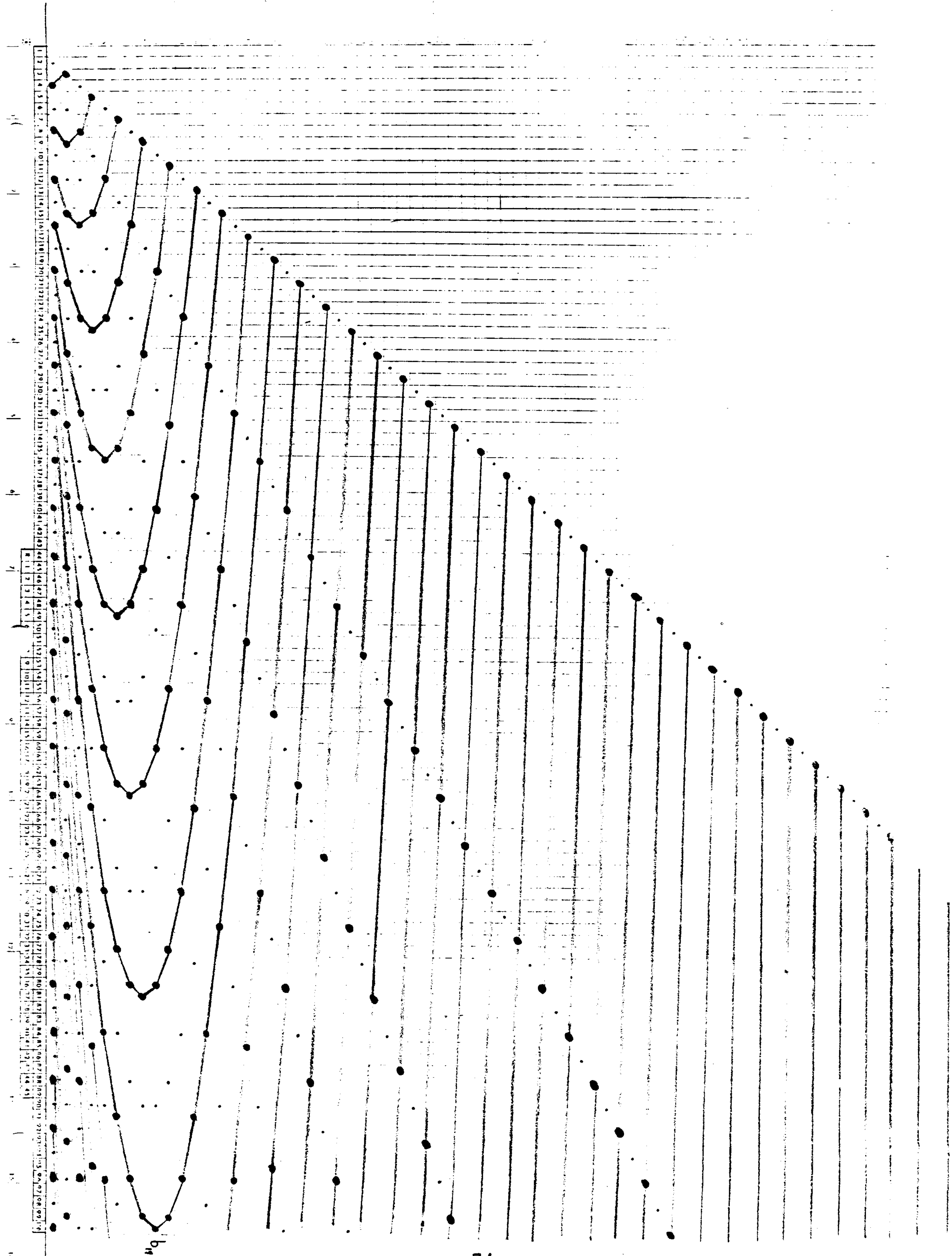

Figure 72.

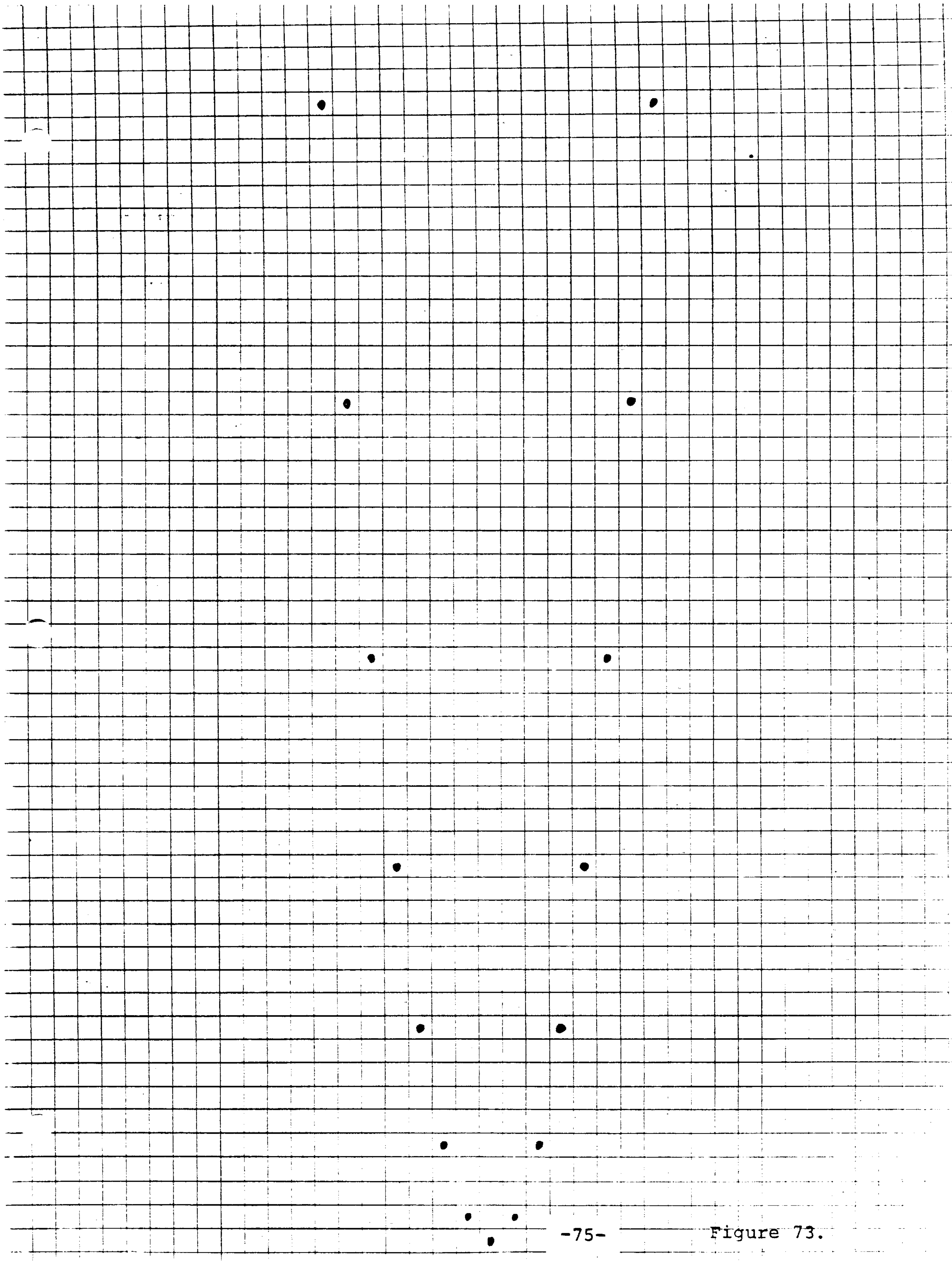

Figure 73.

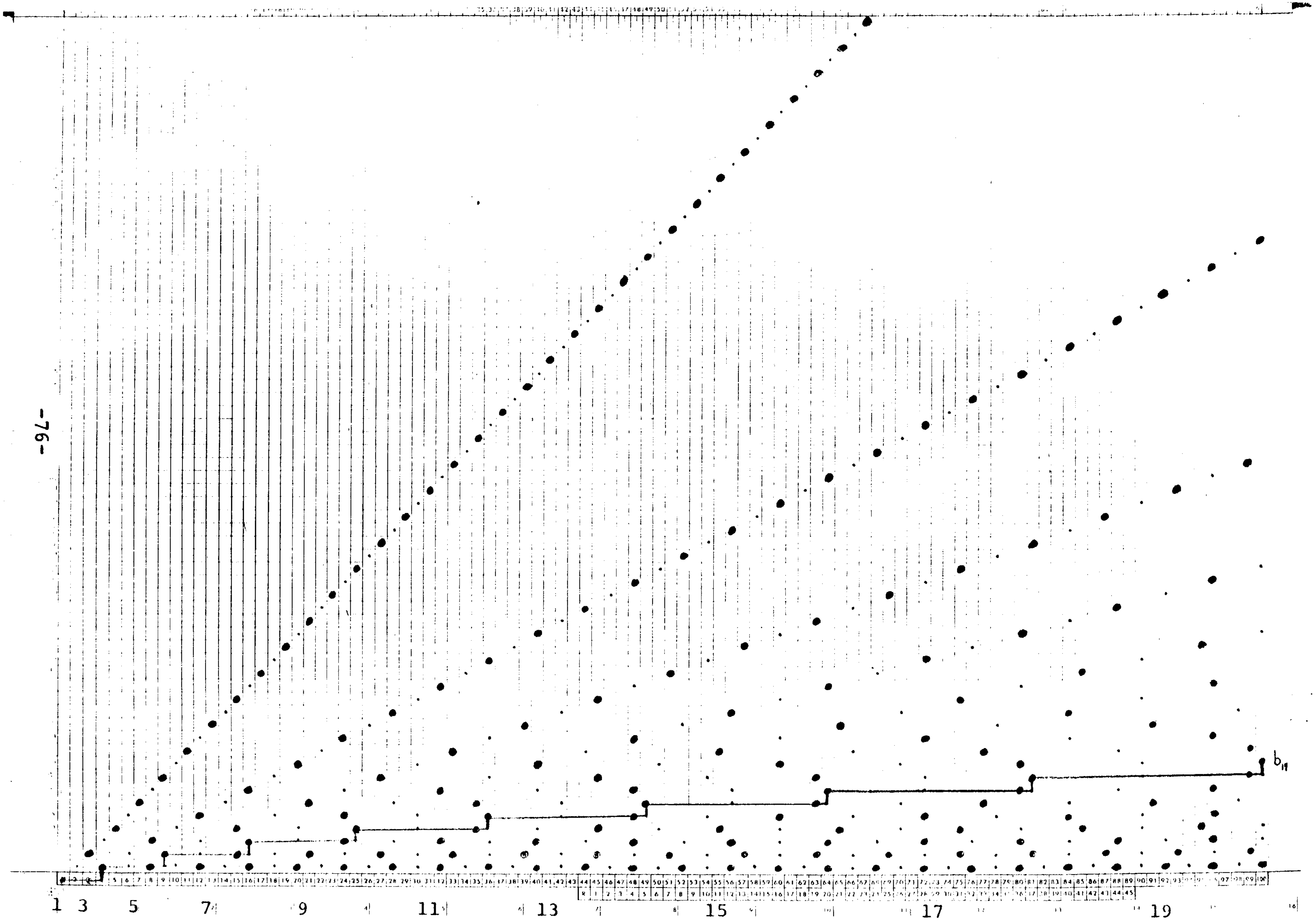

Figure 74.

The spacing of the new void adjacent curves varies somewhat from that of the odd-depositing multiplication table curves. The void curve vertexes are spaced 2, 6, 10, 14, etc., twice the spacing of the table curves (See figure 75.).

As the multiplication curve vertexes are displaced to the right one for each member (the y-value of the vertex gets one higher for each curve), the new right void adjacent curve vertexes are displaced two to the right with each curve (See figure 76.).

It turns out that this right displacement has no bearing on which vertical columns of either graph are empty when they are viewed as explained earlier for the multiplication table and with the void at the right for the new "design". These curves are of the form $x^2 - y^2$ where x^2 is the vertex and y is the dot occurrence left of the vertex (y^2 is the distance left of the vertex of the yth dot) for the multiplication table family. For the new void adjacent families the form is simply $x - y^2$ where x is the vertex and y^2 is the distance of a dot to the left of the vertex.

15. FORMAT AND SPACING --> PRIME KIN?

Perhaps by varying the two parameters governing empty vertical column location (family member form and spacing between successive curve vertexes), various sequences of empty columns, numbered by sequences akin to that of the primes, will be generated.

Perhaps there are more patterns to be found. Who knows what we may find out there in the unexplored approaches to such knowledge? Please be my guest.

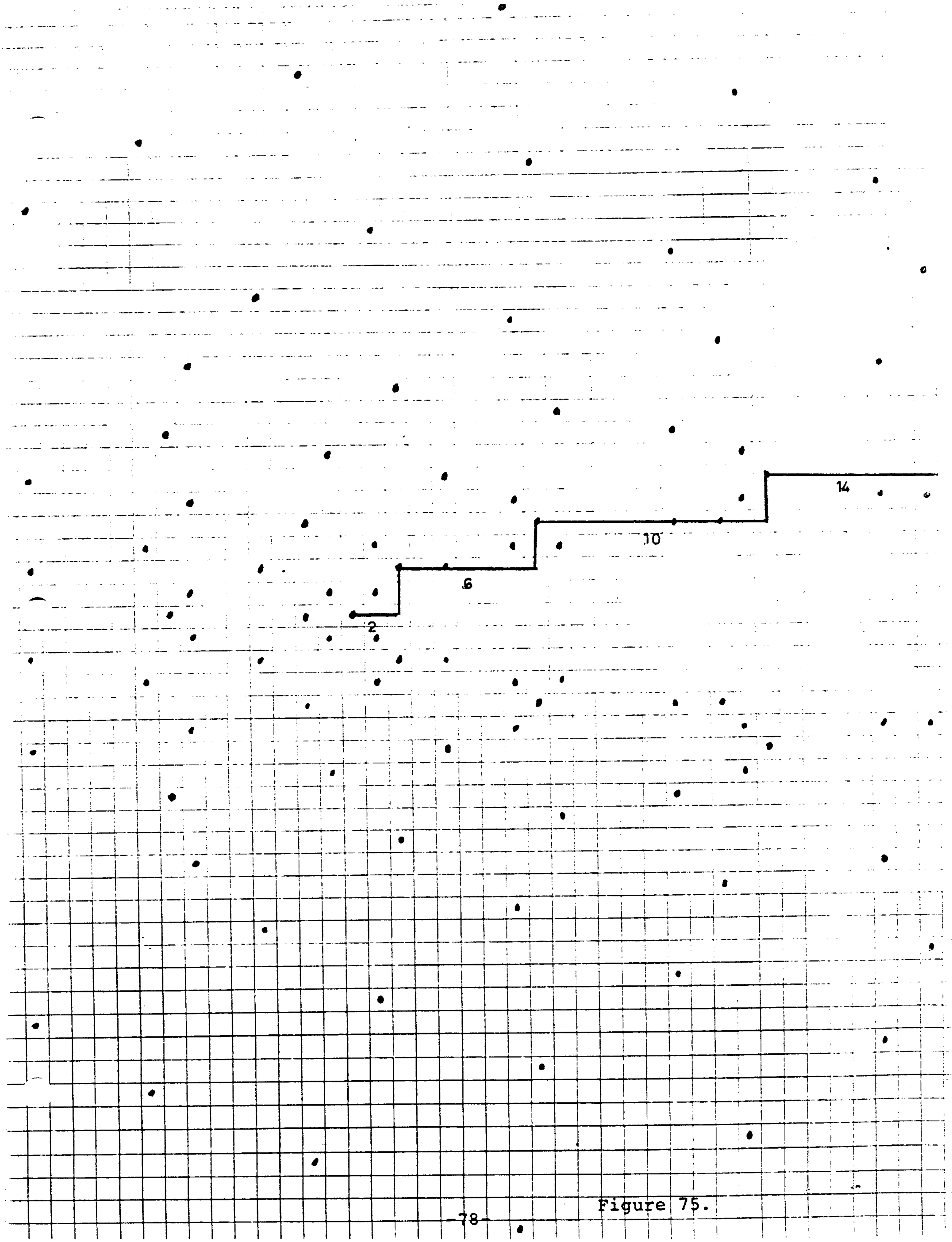

Figure 75.

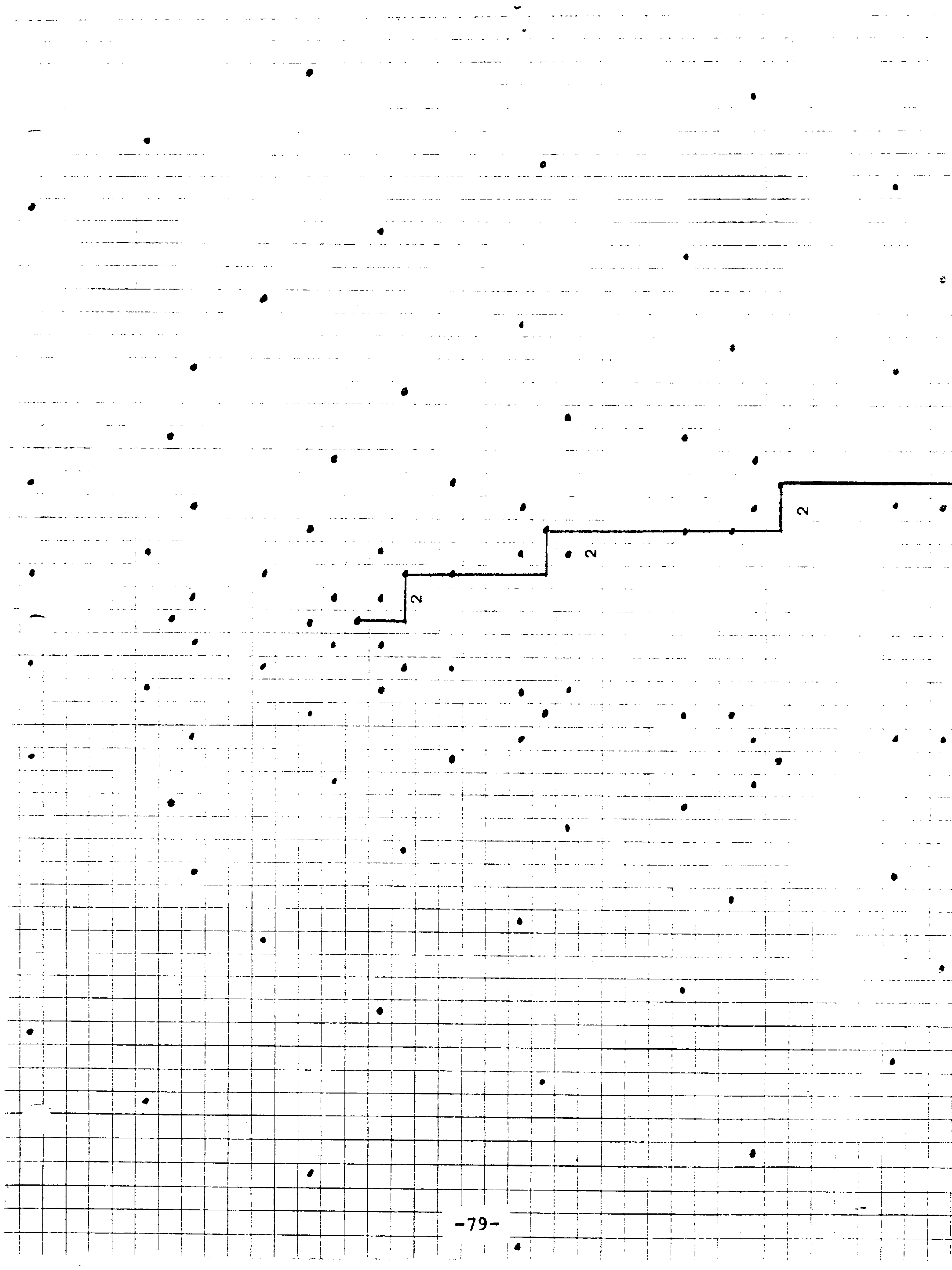

Figure 76.

APPENDIX A

TELLING WHAT NUMBER A DOT IS

How to determine what number is represented by a dot on the triangular array:

Given the triangular array with n spots on line n like the following:

```
1              line 1
2  3           line 2
4  5  6        line 3
  etc.
```

Figure 77.

the coordinates of a spot may be specified as:

Line-number, Spot on the line

or

Dot(x) = Line(y), Spot(z)

When a dot's line-number is determined and its spot number on the line has been counted:

$$\text{Dot}(x) = \frac{L(L-1)}{2} + S \qquad (1)$$

where L = line-number and S = spot number on the line.

For example, the dot in spot one of line 5 is:

$$\frac{5(5-1)}{2} + 1 = 11$$

APPENDIX B

TELLING WHAT SPOT A NUMBER IS

How to tell what line and which spot on it a number is represented at:

Coordinates of x = L,S =

$$L = \left\lceil \frac{\sqrt{8x + 1} - 1}{2} \right\rceil \qquad (2a)$$

$$S = x - \frac{\left\lfloor \frac{\sqrt{8x - 7} - 1}{2} \right\rfloor^2 + \left\lfloor \frac{\sqrt{8x - 7} - 1}{2} \right\rfloor}{2} \qquad (2b)$$

where $\lceil \quad \rceil$ means "rounded up" and $\lfloor \quad \rfloor$ means "rounded down".

For example, to tell which line and spot 11 is located at:

$$L = \left\lceil \frac{\sqrt{8 \times 11 + 1} - 1}{2} \right\rceil = \left\lceil \frac{\sqrt{89} - 1}{2} \right\rceil = \left\lceil \frac{9+ - 1}{2} \right\rceil$$

$$= \left\lceil \frac{8+}{2} \right\rceil = \lceil 4+ \rceil = 5$$

$$S = 11 - \frac{\left[\frac{\sqrt{8 \times 11 - 7} - 1}{2}\right]^2 + \left[\frac{\sqrt{8 \times 11 - 7} - 1}{2}\right]}{2}$$

$$= 11 - \frac{\left[\frac{\sqrt{81} - 1}{2}\right]^2 + \left[\frac{\sqrt{81} - 1}{2}\right]}{2}$$

$$= 11 - \frac{\left[\frac{9 - 1}{2}\right]^2 + \left[\frac{9 - 1}{2}\right]}{2}$$

$$= 11 - \frac{\left[\frac{8}{2}\right]^2 + \left[\frac{8}{2}\right]}{2}$$

$$= 11 - \frac{4^2 + 4}{2}$$

$$= 11 - \frac{20}{2}$$

$$= 11 - 10 = 1$$

So L,S = 5,1.

APPENDIX C

THE DERIVATION OF THE FORMULAS FOR PREDICTING THE CONSTANT HORIZONTAL DISTANCE BETWEEN SUCCESSIVE MEMBER DOTS OF A CURVE IN EACH CURVE FAMILY IN "THE DESIGN"

By inspection of a single page graph of "the design" which has a grid of appropriate lines horizontally and vertically, the first few curve family horizontal constants may be ascertained. By appropriate lines I mean the same with respect to the dots of "the design" as were the grid lines of the graph on which the dots were originally plotted.

The central downward curving curve family contains identical member curves whose constant horizontal change in distance between successive member dots is one unit.

The left and right downward curving curve families both have members whose constant horizontal distance change is two units.

The far left and right downward curving curve families have a horizontal constant of three units. A seemingly simple progression, eh? (See figure 78.)

What about the upward curving families? The left and right void adjacent families have a horizontal constant of one unit, while the far left and right void adjacent curve families have a horizontal constant of four units. By properly sighting along the dots, the third set of left and right void adjacent families may be perceived. This third upward curving family pair has a horizontal constant of seven. (See figure 79.) 1, 4, 7 -- another seemingly simple progression, right?

Well, the fourth set of <u>downward</u> curving curve families has a horizontal constant of 10.

By taking a remote section of "the design", a section to the right of the right edge of the paper when the void is at the bottom (See figure 80.), graphing it onto a very finely lined piece of graph paper (See figure 81.), and extending the patterns in a continuation of "the design", I was able to determine that the fourth set of upward curving families has a horizontal constant of 24. (See figure 82.) (To see how figure 80's points relate to "the design", the upper left corner of figure 80 appears as the rectangle on the lower right in figure 83. The top of figure 83 appears as the larger rectangle in figure 84. That's the very top of figure 80 appearing as the smaller rectangle in figure 84.)

By similar means, I verified that the fifth members of

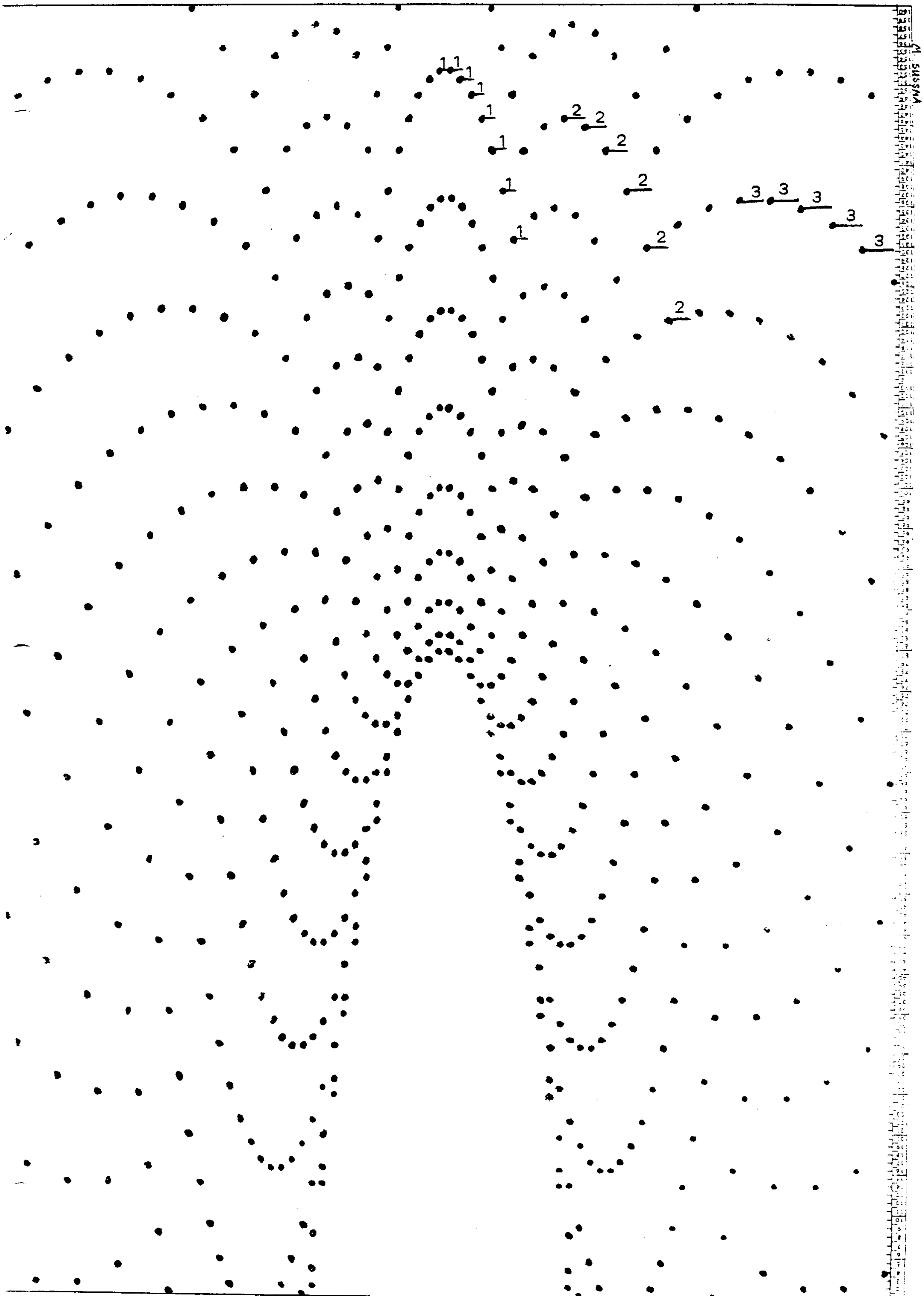

Figure 78.

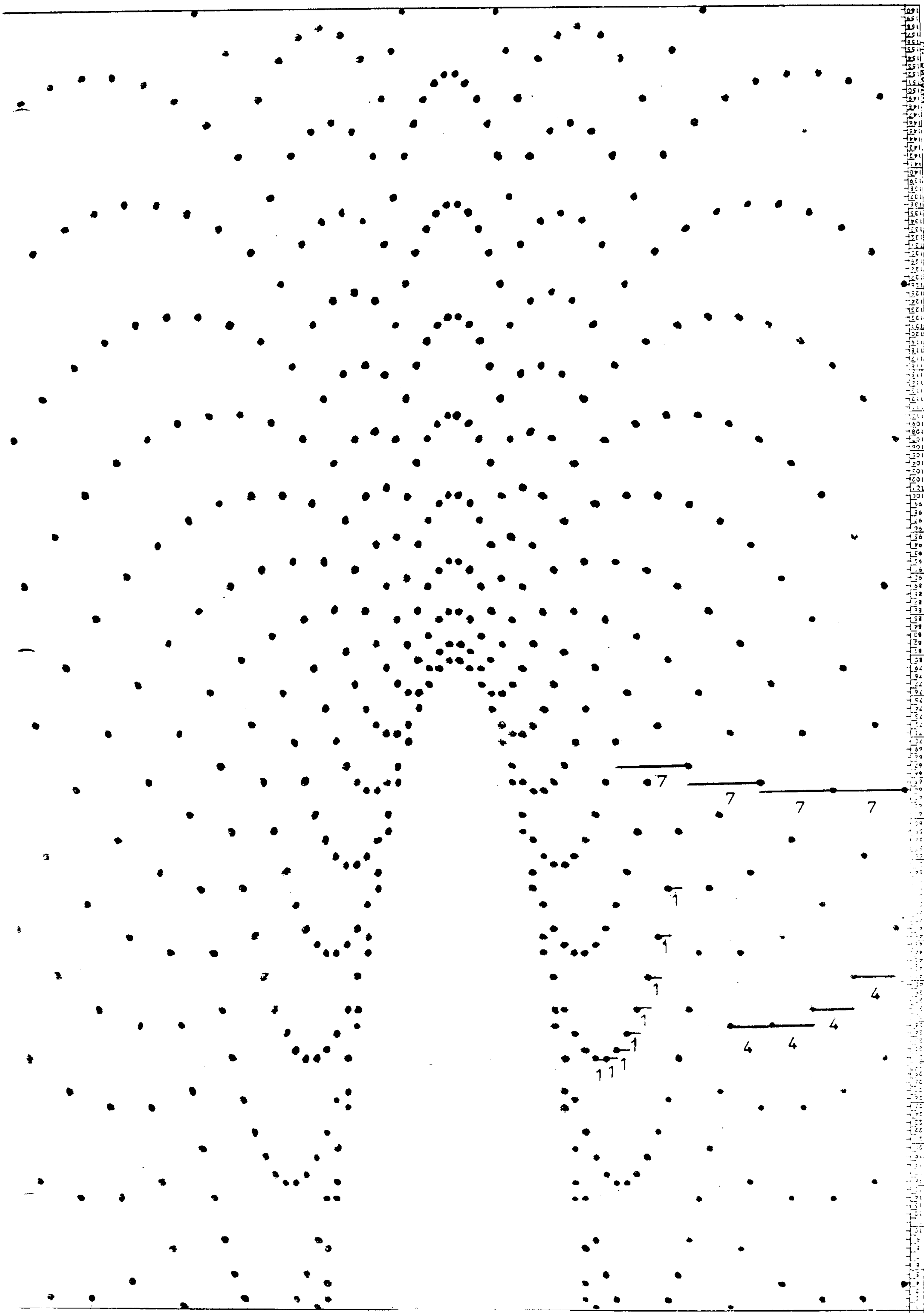

Figure 79.

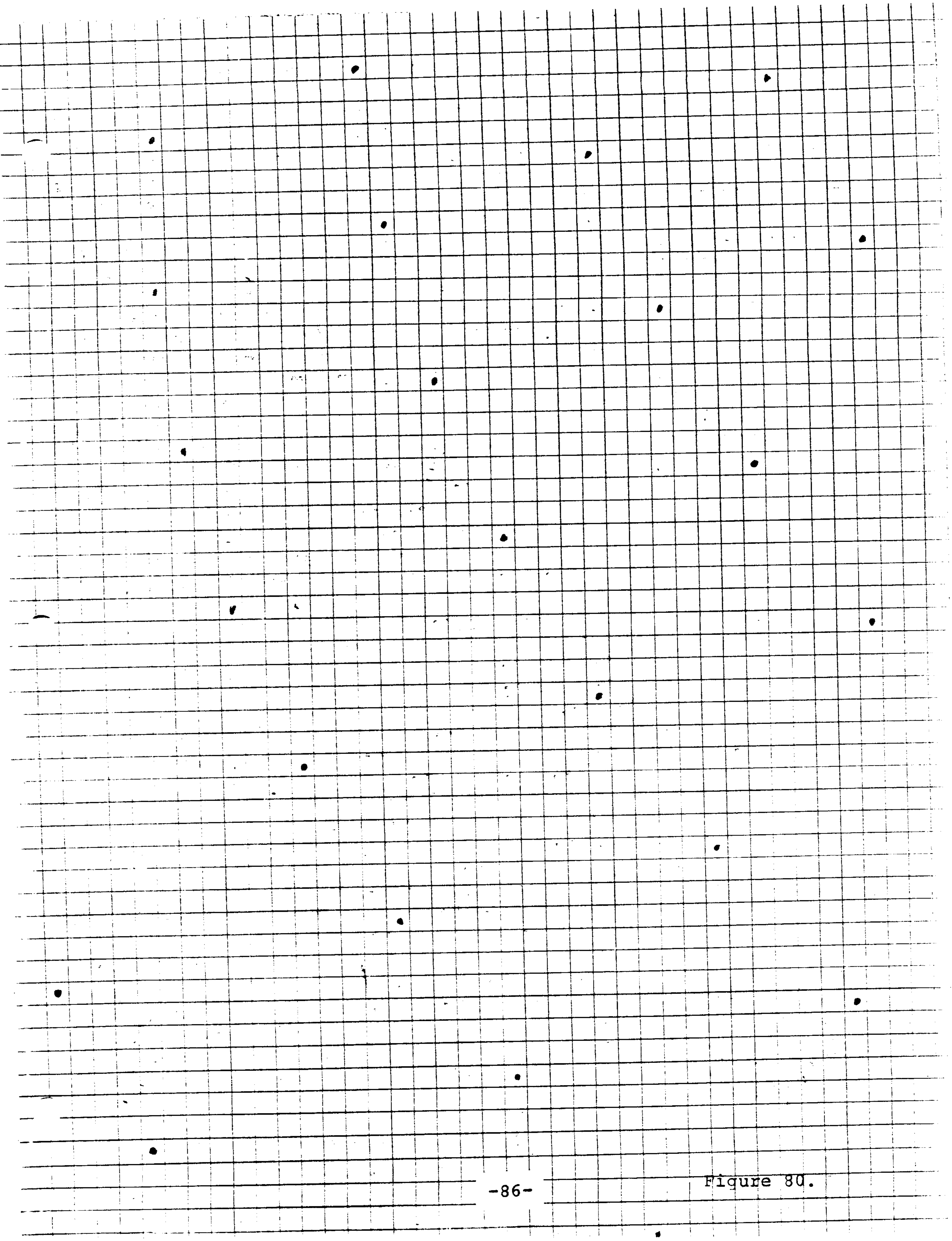

Figure 80.

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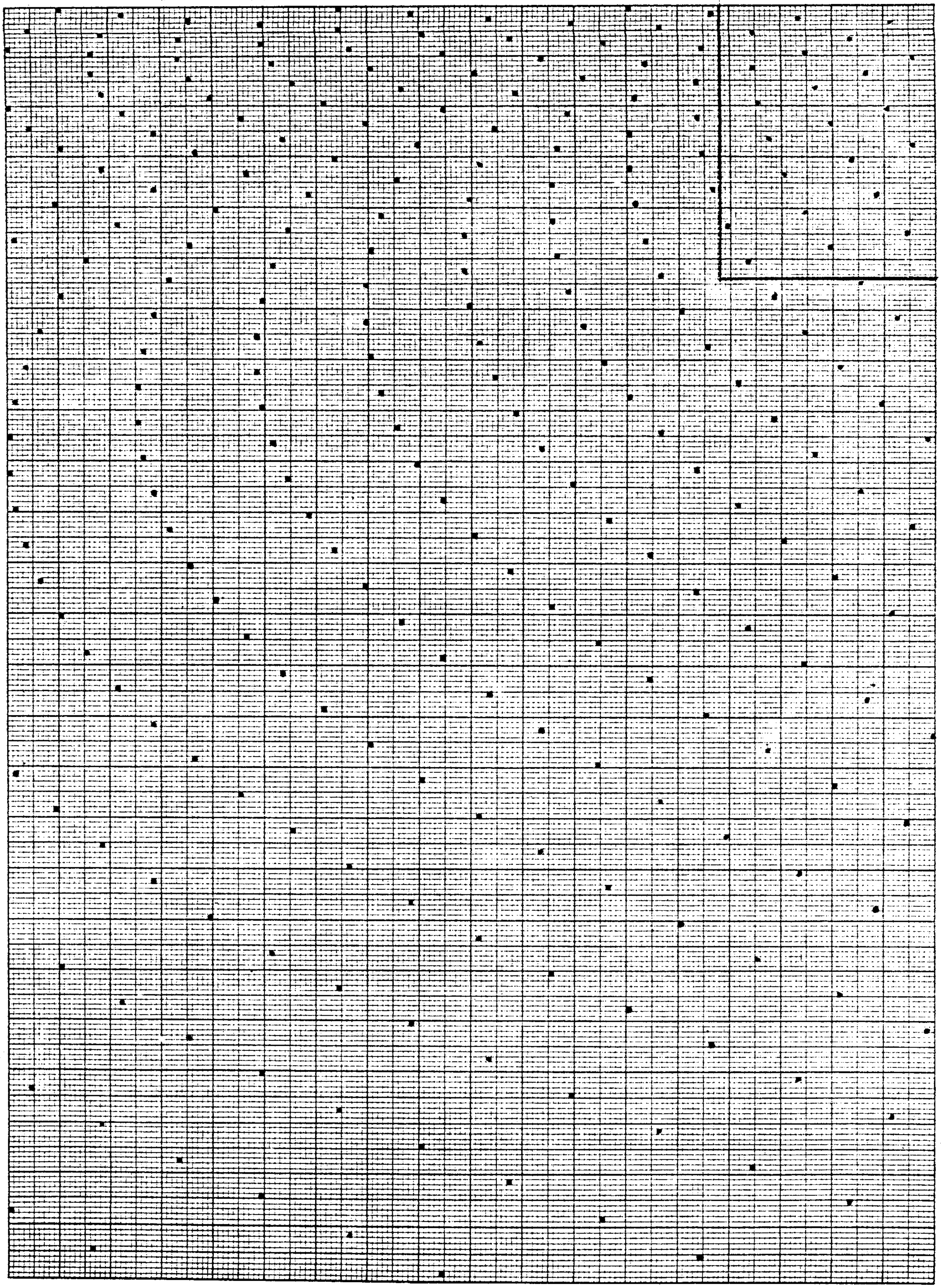
Made in U.S.A.



10 Millimeters to the Centimeter

Figure 81.

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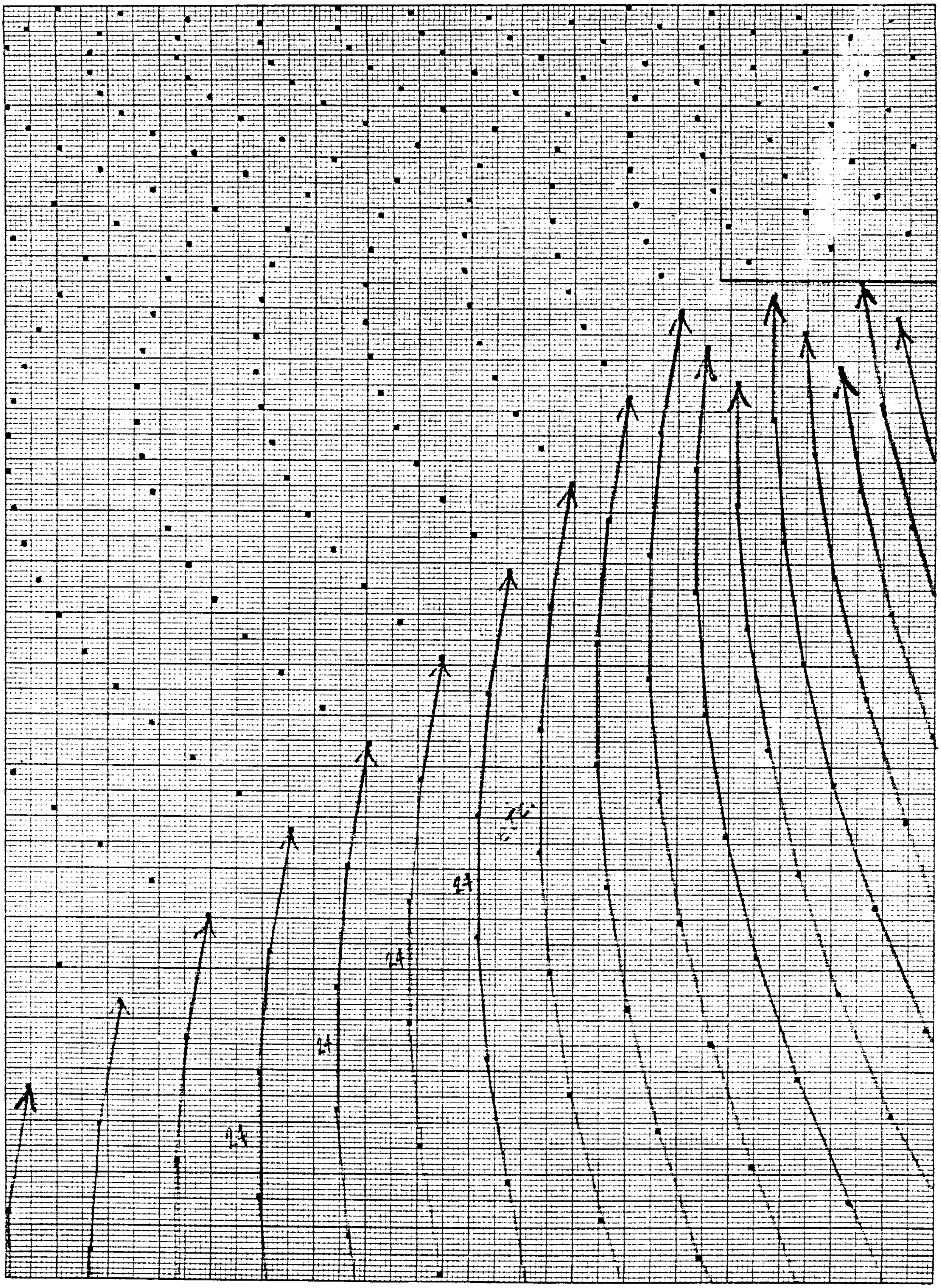


10 Millimeters to the Centimeter

Figure 82.

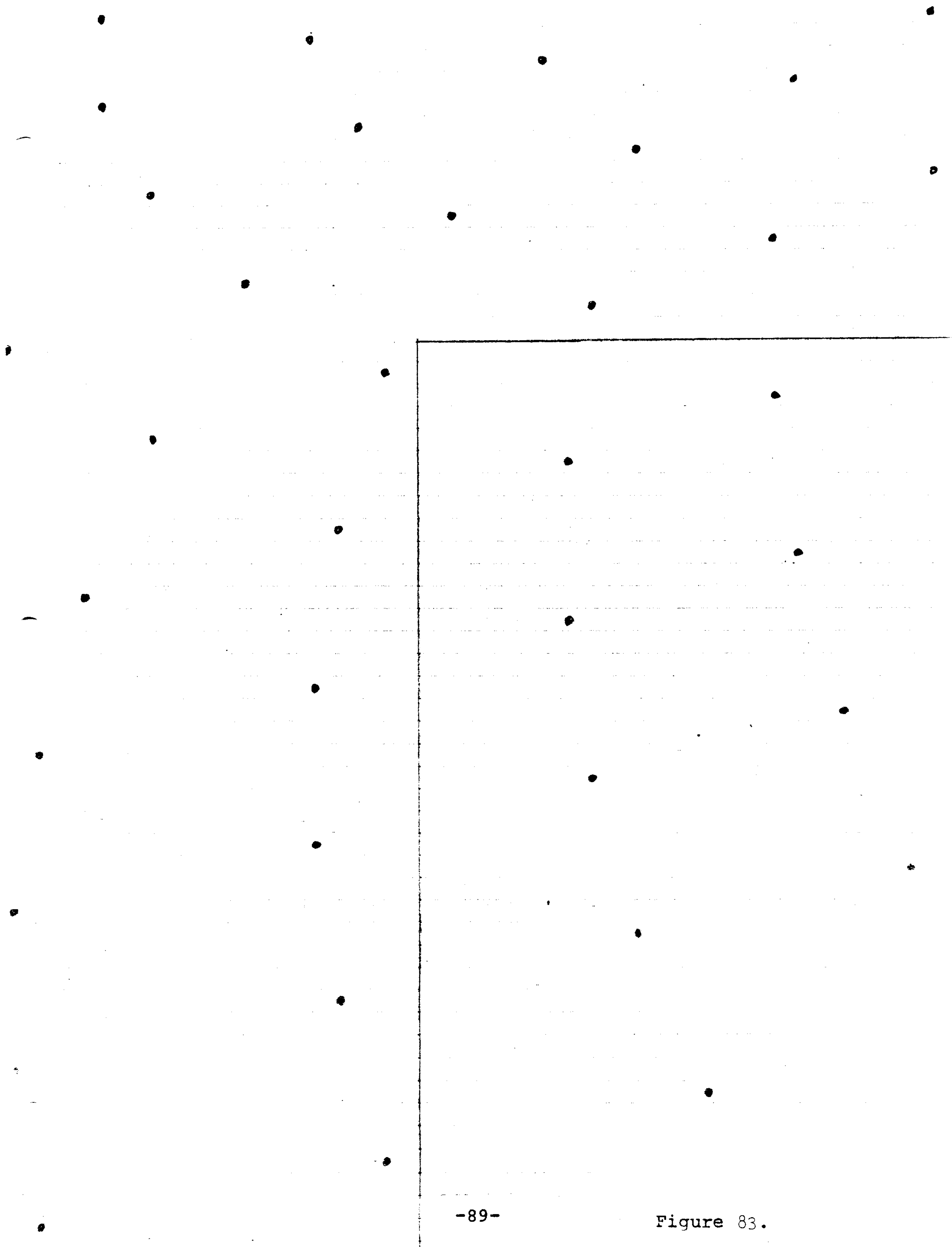


Figure 83.

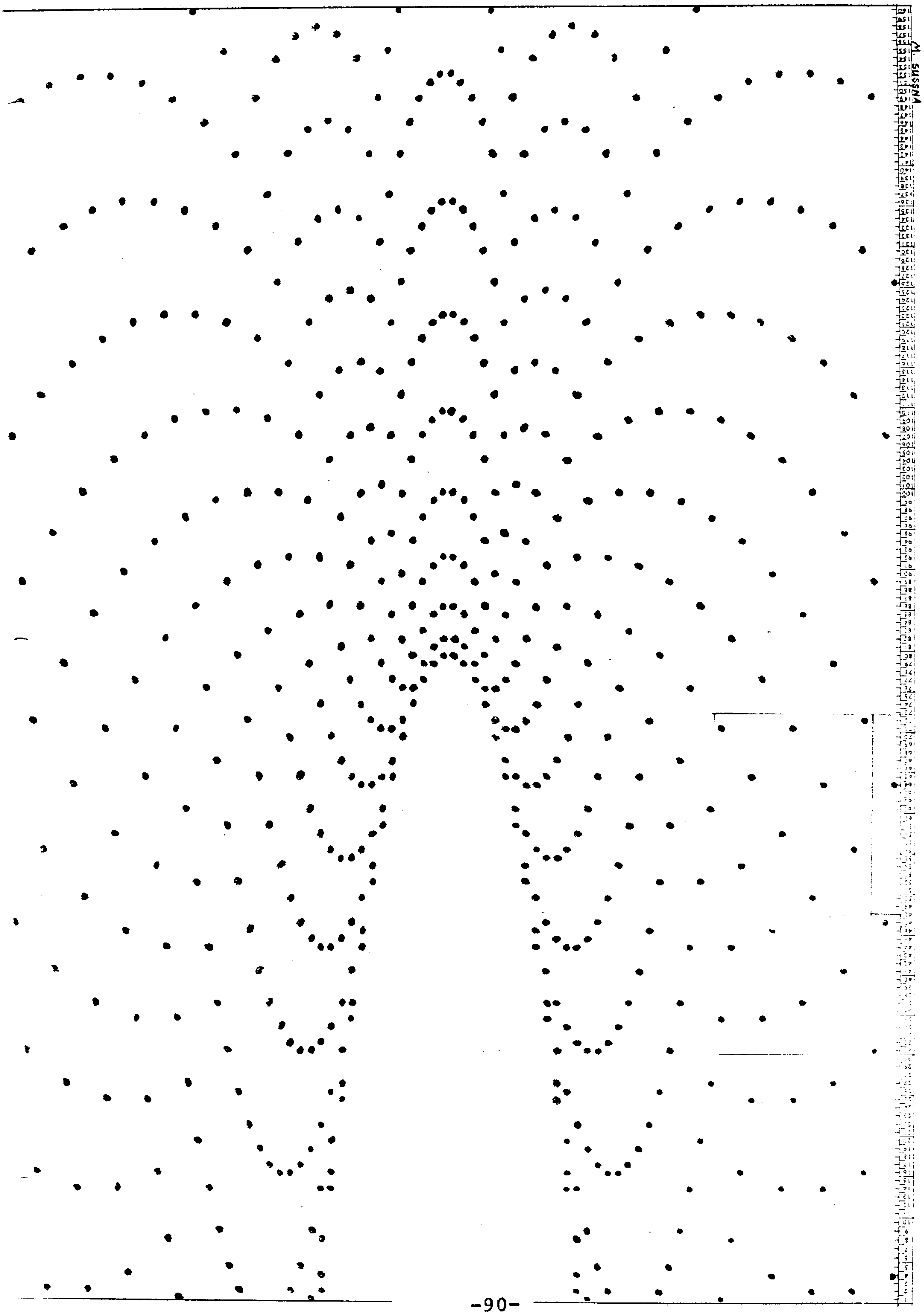


Figure 84.

the two lists are 17 and 41, for downward and upward curving curve families, respectively. (See figures 85 and 86 respectively.)

The list of constants was taking the following shape:

Downward curving	Upward curving
1	1
2	4
3	7
10	24
17	41

Figure 87.

With this much data it became possible to begin to see the pattern in these constants:

D		U	
1		1	
	1		3
2		4	
	1		3
3		7	
	7		17
10		24	
	7		17
17		41	

Figure 88.

The differences come in pairs - two ones, two threes, two sevens, and two 17's. If that progression of differences (1, 3, 7, 17) could be "cracked", then perhaps the once simple seeming constant lists could still be deciphered.

By rewriting the pair of lists offset by one, a neat effect occurs:

D	U	D+U
1	0	1
2	1	3
3	4	7
10	7	17
17	24	41
	41	

Figure 89.

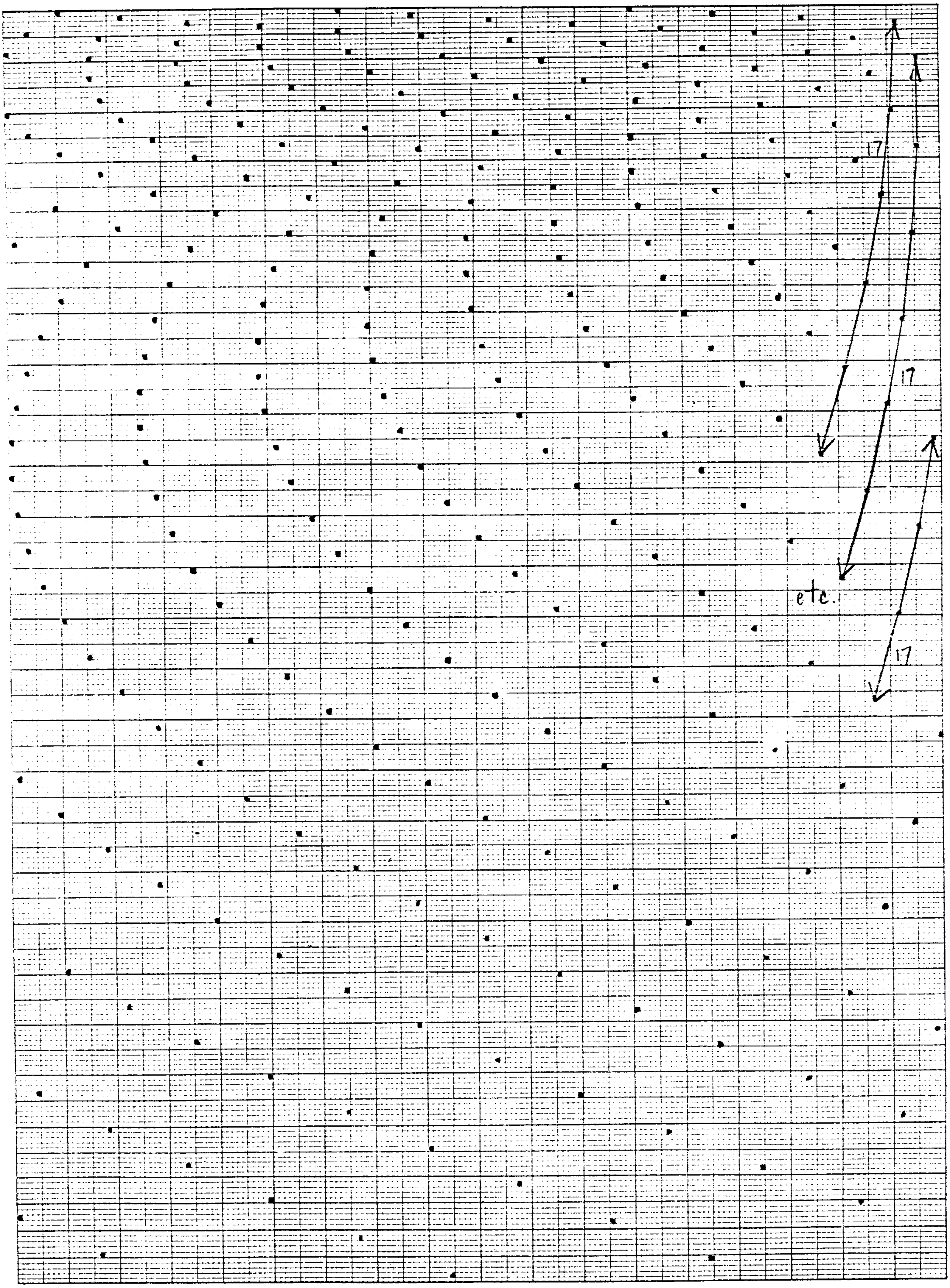


Figure 85.

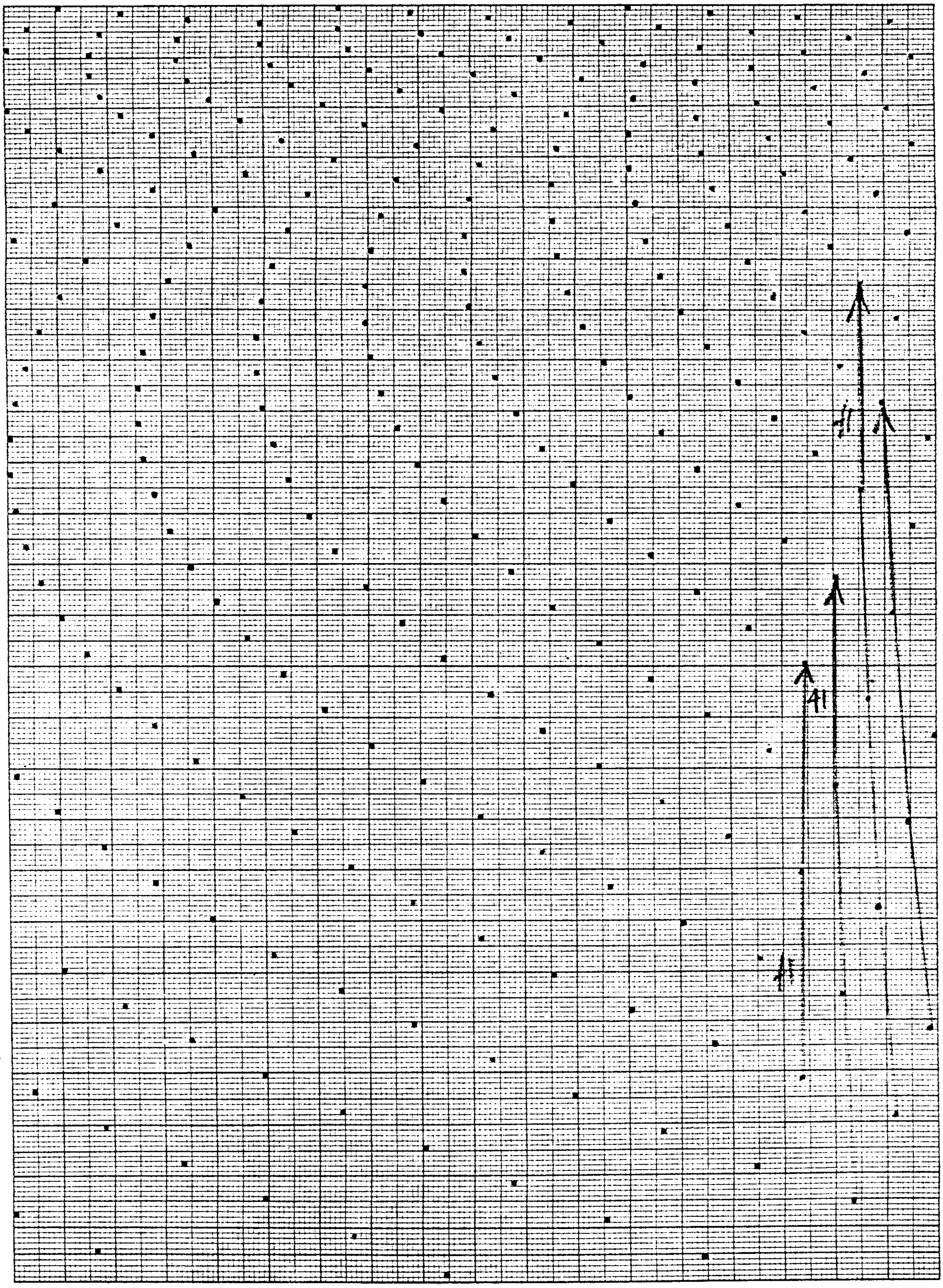


Figure 86.

Our difference progression 1, 3, 7, 17 is the sum across the pair of lists for the first four entries of the pair:

$$D(n) + U(n-1) = Diff(n) \qquad (3)$$

The five sums on the right all appear as individual members of the list pair in this way:

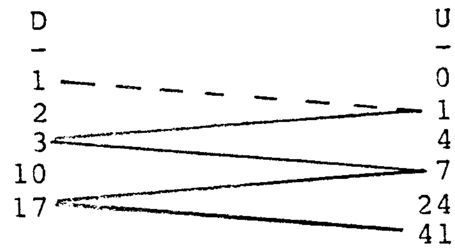


Figure 90.

Should the two 1's connected by the dotted line be connnected? For the first 1 in the D list to be admitted to the sequence 1, 3, 7, 17 it should be a difference pair in the U list in the following way:

```
U
-
-1
    1
 0
    1
 1
    3
 4
    3
 7
    17
24
    17
41
```

Figure 91.

By the same token, 41 should be the next difference pair in the D list thus:

D	
1	
	1
2	
	1
3	
	7
10	
	7
17	
	41
58	
	41
99	

Figure 92.

Is there any way to check the truth of these extrapolations? Yes. There should be detectable left and right downward curving curve families with a constant horizontal change in distance between successive dots in a member curve equal to these extrapolated values. There are. First, in the U list, curves with a 0 constant are "in" the void, and curves with a constant of -1 correspond to the left void adjacent curves, if we started on the right of the void. Downward curving curves with a horizontal constant of 58 can be seen in figure 93.

Now, to get a feel for what type of formula the U and D lists might have, I recognized the resemblance between the difference progression 1, 1, 3, 7, 17, 41, ... and the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, and so on. While each Fibonacci number is the sum of the two preceding Fibonacci numbers, the difference numbers are the sum of the immediately preceding number doubled and the number preceding that:

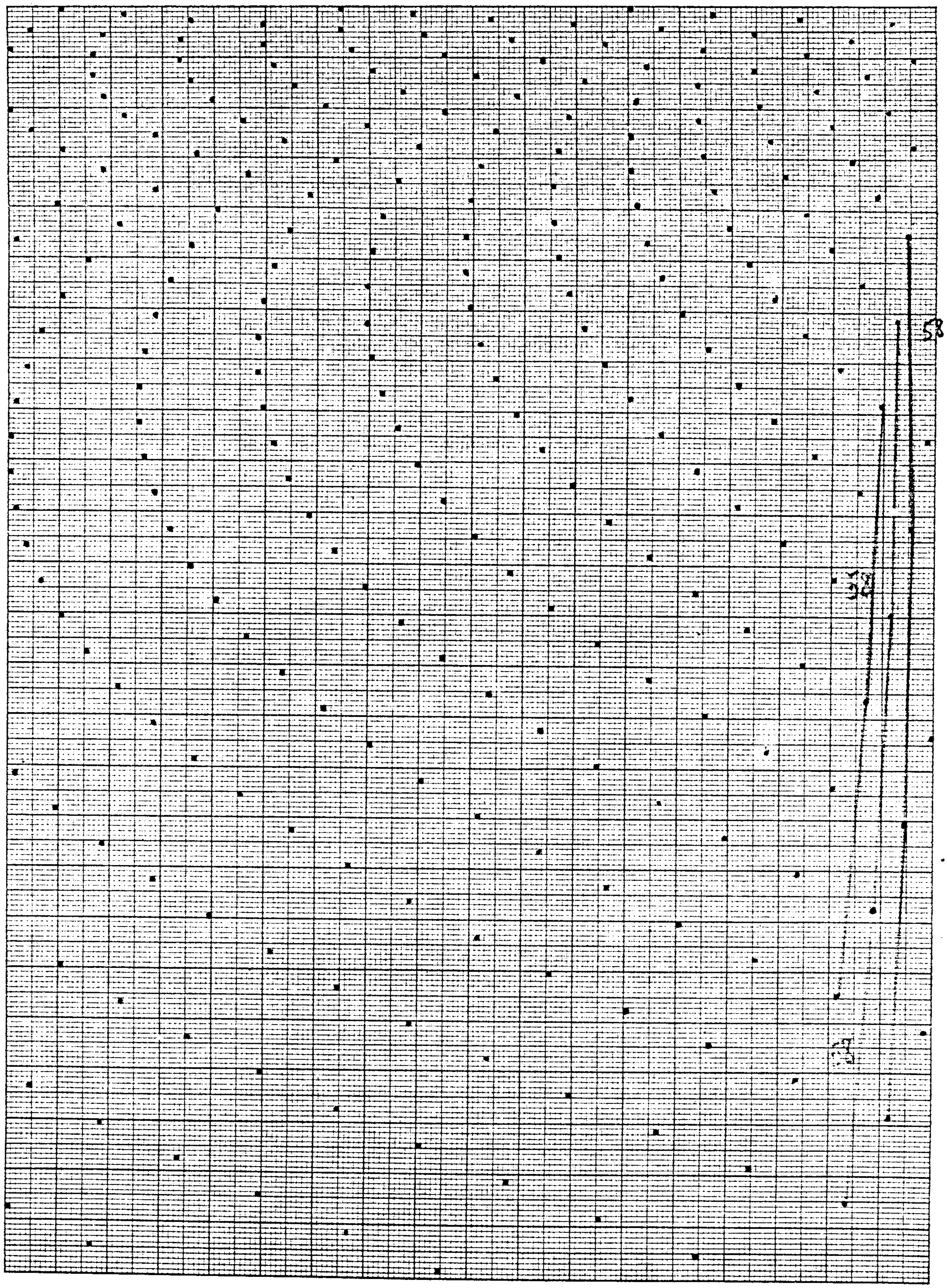
$$\mathrm{Fibo}(n) = \mathrm{Fibo}(n-1) + \mathrm{Fibo}(n-2) \qquad (4)$$

$$\mathrm{Diff}(n) = 2 \times \mathrm{Diff}(n-1) + \mathrm{Diff}(n-2) \qquad (5)$$

Perhaps the formula for the difference numbers would resemble the formula for the Fibonacci numbers:

$$\mathrm{Fibo}(n) = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right] \qquad (6)$$

The key to such a formula is the term being raised to the nth power.



10 Millimeters to the Centimeter

Figure 93.

For example $(1 + \sqrt{5})^n$ goes through interesting gyrations from power to power. The resulting coefficients contain the seeds of the Fibonacci numbers:

$$(1 + \sqrt{5})^0 = 1 \tag{7}$$

$$(1 + \sqrt{5})^1 = 1 + \sqrt{5} \tag{8}$$

$$(1 + \sqrt{5})^2 = 6 + 2\sqrt{5} \tag{9}$$

$$(1 + \sqrt{5})^3 = 16 + 8\sqrt{5} \tag{10}$$

$$(1 + \sqrt{5})^4 = 56 + 24\sqrt{5} \tag{11}$$

$$(1 + \sqrt{5})^5 = 176 + 80\sqrt{5} \tag{12}$$

and so on.

The multiples of $\sqrt{5}$ in each result form a pattern:

1
2
8
24
80

Figure 94.

When these numbers are doubled, we get:

2
4
16
48
160

Figure 95.

When these numbers are divided by successive powers of two we get the Fibonacci numbers:

2 / 2 = 1
4 / 4 = 1
16 / 8 = 2
48 / 16 = 3
160 / 32 = 5

Figure 96.

The rest of the items in the formula for the Fibonacci numbers besides $(1 + \sqrt{5})^n$ are merely what is needed to isolate those key multiples mentioned above and scale them down properly (double them and divide the result by the appropriate power of two).

It was with this in mind that I set out to conquer the difference progression of numbers.

I first tried variations on $(1 + \sqrt{5})^n$ like $(2 + \sqrt{5})^n$ and $(1 + 2\sqrt{5})^n$ but the sequences of coefficients grew too rapidly.

Next I tried $(1 + \sqrt{3})^n$. This was better but still not right.

With my next try, I struck it rich:

$$(1 + \sqrt{2})^0 = 1 \qquad (13)$$

$$(1 + \sqrt{2})^1 = 1 + \sqrt{2} \qquad (14)$$

$$(1 + \sqrt{2})^2 = 3 + 2\sqrt{2} \qquad (15)$$

$$(1 + \sqrt{2})^3 = 7 + 5\sqrt{2} \qquad (16)$$

$$(1 + \sqrt{2})^4 = 17 + 12\sqrt{2} \qquad (17)$$

$$(1 + \sqrt{2})^5 = 41 + 29\sqrt{2} \qquad (18)$$

and so on.

This time we wish to isolate the coefficients of the non-square-root part of the result when the key term is raised to the nth power. This is done easily enough:

$$1 = \frac{(1 + \sqrt{2}) + (1 - \sqrt{2})}{2} \qquad (19)$$

$$3 = \frac{(3 + 2\sqrt{2}) + (3 - 2\sqrt{2})}{2} \qquad (20)$$

etc.

It so happens that $(1 - \sqrt{2})^n$ looks exactly like

$(1+\sqrt{2})^n$ when multiplied out, except for the sign between the non-$\sqrt{2}$ part and the $\sqrt{2}$ part of the result.

For example,

$$(1+\sqrt{2})^4 = 17 + 12\sqrt{2}$$

while

$$(1-\sqrt{2})^4 = 17 - 12\sqrt{2}$$

and so on.

So, difference number n equals

$$\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2} . \qquad (21)$$

Now, how can the D and U lists be expressed based on the preceding find? We start with the following table:

n	D	U	Diff
1	1	1	1
2	2	4	3
3	3	7	7
4	10	24	17
5	17	41	41

Figure 97.

It would seem that the odd D's and U's correspond to the difference numbers:

D(1) = Diff(0)	U(1) = Diff(1)
D(3) = Diff(2)	U(3) = Diff(3)
D(5) = Diff(4)	U(5) = Diff(5)

Figure 98.

So D(n) = Diff(n-1) when n is odd, and U(n) = Diff(n) when n is odd.

What about when n is even? We'll need our earlier finding that D(n) + U(n-1) = Diff(n).

By rearranging terms we get:

$$D(n) = Diff(n) - U(n-1). \quad (22)$$

Now if n is even, an even D, D(n), equals Diff(n) minus U(n-1), an odd U. We know U(n) = Diff(n) for odd n's, so

$$U(n-1) = Diff(n-1). \quad (23)$$

We now have D(n) = Diff(n) - Diff(n-1) when n is even.

All that's left to express in terms of the difference progression is U(n) when n is even.

We again call on equation 3:

$$D(n) + U(n-1) = Diff(n)$$

This time we rearrange the terms this way:

$$U(n-1) = Diff(n) - D(n). \quad (24)$$

We want an even U, so n-1 is even. That means that n is odd.

D(n) = Diff(n-1) for odd n, so, for even n:

$$U(n-1) = Diff(n) - Diff(n-1) \quad (25)$$

or

$$U(n) = Diff(n+1) - Diff(n). \quad (26)$$

For example:

$$U(4) = Diff(5) - Diff(4)$$

or

$$24 = 41 - 17.$$

To summarize:

	D(n)	U(n)
ODD	Diff(n-1)	Diff(n)
EVEN	Diff(n) - Diff(n-1)	Diff(n+1) - Diff(n)

Figure 99.

or

	D(n)	U(n)
ODD	$\frac{(1+\sqrt{2})^{n-1} + (1-\sqrt{2})^{n-1}}{2}$	$\frac{(1+\sqrt{2})^{n} + (1-\sqrt{2})^{n}}{2}$
EVEN	$\frac{(1+\sqrt{2})^{n} + (1-\sqrt{2})^{n}}{2} - \frac{(1+\sqrt{2})^{n-1} + (1-\sqrt{2})^{n-1}}{2}$	$\frac{(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}}{2} - \frac{(1+\sqrt{2})^{n} + (1-\sqrt{2})^{n}}{2}$

Figure 100.

The complicated expressions for D(n) and U(n) for even n's may be simplified in the following way:

$$D(n) = \frac{(1+\sqrt{2})^{n} + (1-\sqrt{2})^{n}}{2} - \frac{(1+\sqrt{2})^{n-1} + (1-\sqrt{2})^{n-1}}{2} \quad (27)$$

$$= \frac{(1+\sqrt{2})^{n} + (1-\sqrt{2})^{n} - (1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}}{2}$$

$$= \frac{(1+\sqrt{2})^{n} - (1+\sqrt{2})^{n-1} + (1-\sqrt{2})^{n} - (1-\sqrt{2})^{n-1}}{2}$$

$$= \frac{[(1+\sqrt{2})^{n} - (1+\sqrt{2})^{n-1}] + [(1-\sqrt{2})^{n} - (1-\sqrt{2})^{n-1}]}{2}$$

$$= \frac{[(1+\sqrt{2})(1+\sqrt{2})^{n-1} - (1+\sqrt{2})^{n-1}] + [(1-\sqrt{2})(1-\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}]}{2}$$

$$= \frac{(1+\sqrt{2})^{n-1}[(1+\sqrt{2}) - 1] + (1-\sqrt{2})^{n-1}[(1-\sqrt{2}) - 1]}{2}$$

$$= \frac{[(1+\sqrt{2})^{n-1}(1+\sqrt{2}-1)] + [(1-\sqrt{2})^{n-1}(1-\sqrt{2}-1)]}{2}$$

$$= \frac{[(1+\sqrt{2})^{n-1}(\sqrt{2})] - [(1-\sqrt{2})^{n-1}(\sqrt{2})]}{2}$$

$$= \frac{\sqrt{2}[(1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}]}{2}$$

So, D(n) for even n =

$$\frac{(1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}}{\sqrt{2}} \qquad (28)$$

By the same procedure for U(n) for even n's, where

$$U(n) = \frac{(1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}}{2} - \frac{(1+\sqrt{2})^{n} + (1-\sqrt{2})^{n}}{2} \qquad (29)$$

we arrive at

$$U(n) = \frac{(1+\sqrt{2})^{n} - (1-\sqrt{2})^{n}}{\sqrt{2}} \qquad (30)$$

for even n.

Here is the revised summary:

	D(n)	U(n)
ODD	$\frac{(1+\sqrt{2})^{n-1} + (1-\sqrt{2})^{n-1}}{2}$	$\frac{(1+\sqrt{2})^{n} + (1-\sqrt{2})^{n}}{2}$
EVEN	$\frac{(1+\sqrt{2})^{n-1} - (1-\sqrt{2})^{n-1}}{\sqrt{2}}$	$\frac{(1+\sqrt{2})^{n} - (1-\sqrt{2})^{n}}{\sqrt{2}}$

Figure 101.

APPENDIX D

PRIMES GENERATED BY THE NINE FORMULAS

x	1	2	3	4	5	6	7	8	9
0	5	5	7	11	11	17	19	29	41
1	7	7	11	13	13	19	23	31	43
2	11	13	19	17	19	23	31	37	47
3	17	23	31	23	29	29	43	47	53
4	-	37	47	31	43	37	59	61	61
5	-	-	67	41	61	47	79	79	71
6	-	-	-	53	83	59	103	101	83
7	-	-	-	67	109	73	131	127	97
8	-	-	-	83	139	89	163	157	113
9	-	-	-	101	173	107	199	191	131
10	-	-	-	-	211	127	239	229	151
11	-	-	-	-	-	149	283	271	173
12	-	-	-	-	-	173	331	317	197
13	-	-	-	-	-	199	383	367	223
14	-	-	-	-	-	227	439	421	251
15	-	-	-	-	-	257	499	479	281
16	-	-	-	-	-	-	563	541	313
17	-	-	-	-	-	-	631	607	347
18	-	-	-	-	-	-	-	677	383
19	-	-	-	-	-	-	-	751	421
20	-	-	-	-	-	-	-	829	461
21	-	-	-	-	-	-	-	911	503
22	-	-	-	-	-	-	-	997	547
23	-	-	-	-	-	-	-	1087	593
24	-	-	-	-	-	-	-	1181	641
25	-	-	-	-	-	-	-	1279	691
26	-	-	-	-	-	-	-	1381	743
27	-	-	-	-	-	-	-	1487	797
28	-	-	-	-	-	-	-	1597	853
29	-	-	-	-	-	-	-	-	911
30	-	-	-	-	-	-	-	-	971
31	-	-	-	-	-	-	-	-	1033
32	-	-	-	-	-	-	-	-	1097
33	-	-	-	-	-	-	-	-	1163
34	-	-	-	-	-	-	-	-	1231
35	-	-	-	-	-	-	-	-	1301
36	-	-	-	-	-	-	-	-	1373
37	-	-	-	-	-	-	-	-	1447
38	-	-	-	-	-	-	-	-	1523
39	-	-	-	-	-	-	-	-	1601

Legend for Appendix D

column	formula
1	$x^2 + x + 5$
2	$2x^2 + 5$
3	$2x^2 + 2x + 7$
4	$x^2 + x + 11$
5	$2x^2 + 11$
6	$x^2 + x + 17$
7	$2x^2 + 2x + 19$
8	$2x^2 + 29$
9	$x^2 + x + 41$

APPENDIX E

TABLE OF PRIMES TO 8911

2	197	461	751	1051	1381	1697	2039	2381	2719
3	199	463	757	1061	1399	1699	2053	2383	2729
5	211	467	761	1063	1409	1709	2063	2389	2731
7	223	479	769	1069	1423	1721	2069	2393	2741
11	227	487	773	1087	1427	1723	2081	2399	2749
13	229	491	787	1091	1429	1733	2083	2411	2753
17	233	499	797	1093	1433	1741	2087	2417	2767
19	239	503	809	1097	1439	1747	2089	2423	2777
23	241	509	811	1103	1447	1753	2099	2437	2789
29	251	521	821	1109	1451	1759	2111	2441	2791
31	257	523	823	1117	1453	1777	2113	2447	2797
37	263	541	827	1123	1459	1783	2129	2459	2801
41	269	547	829	1129	1471	1787	2131	2467	2803
43	271	557	839	1151	1481	1789	2137	2473	2819
47	277	563	853	1153	1483	1801	2141	2477	2833
53	281	569	857	1163	1487	1811	2143	2503	2837
59	283	571	859	1171	1489	1823	2153	2521	2843
61	293	577	863	1181	1493	1831	2161	2531	2851
67	307	587	877	1187	1499	1847	2179	2539	2857
71	311	593	881	1193	1511	1861	2203	2543	2861
73	313	599	883	1201	1523	1867	2207	2549	2879
79	317	601	887	1213	1531	1871	2213	2551	2887
83	331	607	907	1217	1543	1873	2221	2557	2897
89	337	613	911	1223	1549	1877	2237	2579	2903
97	347	617	919	1229	1553	1879	2239	2591	2909
101	349	619	929	1231	1559	1889	2243	2593	2917
103	353	631	937	1237	1567	1901	2251	2609	2927
107	359	641	941	1249	1571	1907	2267	2617	2939
109	367	643	947	1259	1579	1913	2269	2621	2953
113	373	647	953	1277	1583	1931	2273	2633	2957
127	379	653	967	1279	1597	1933	2281	2647	2963
131	383	659	971	1283	1601	1949	2287	2657	2969
137	389	661	977	1289	1607	1951	2293	2659	2971
139	397	673	983	1291	1609	1973	2297	2663	2999
149	401	677	991	1297	1613	1979	2309	2671	3001
151	409	683	997	1301	1619	1987	2311	2677	3011
157	419	691	1009	1303	1621	1993	2333	2683	3019
163	421	701	1013	1307	1627	1997	2339	2687	3023
167	431	709	1019	1319	1637	1999	2341	2689	3037
173	433	719	1021	1321	1657	2003	2347	2693	3041
179	439	727	1031	1327	1663	2011	2351	2699	3049
181	443	733	1033	1361	1667	2017	2357	2707	3061
191	449	739	1039	1367	1669	2027	2371	2711	3067
193	457	743	1049	1373	1693	2029	2377	2713	3079

3083	3463	3803	4159	4549	4943	5323	5689	6079	6449
3089	3467	3821	4177	4561	4951	5333	5693	6089	6451
3109	3469	3823	4201	4567	4957	5347	5701	6091	6469
3119	3491	3833	4211	4583	4967	5351	5711	6101	6473
3121	3499	3847	4217	4591	4969	5381	5717	6113	6481
3137	3511	3851	4219	4597	4973	5387	5737	6121	6491
3163	3517	3853	4229	4603	4987	5393	5741	6131	6521
3167	3527	3863	4231	4621	4993	5399	5743	6133	6529
3169	3529	3877	4241	4637	4999	5407	5749	6143	6547
3181	3533	3881	4243	4639	5003	5413	5779	6151	6551
3187	3539	3889	4253	4643	5009	5417	5783	6163	6553
3191	3541	3907	4259	4649	5011	5419	5791	6173	6563
3203	3547	3911	4261	4651	5021	5431	5801	6197	6569
3209	3557	3917	4271	4657	5023	5437	5807	6199	6571
3217	3559	3919	4273	4663	5039	5441	5813	6203	6577
3221	3571	3923	4283	4673	5051	5443	5821	6211	6581
3229	3581	3929	4289	4679	5059	5449	5827	6217	6599
3251	3583	3931	4297	4691	5077	5471	5839	6221	6607
3253	3593	3943	4327	4703	5081	5477	5843	6229	6619
3257	3607	3947	4337	4721	5087	5479	5849	6247	6637
3259	3613	3967	4339	4723	5099	5483	5851	6257	6653
3271	3617	3989	4349	4729	5101	5501	5857	6263	6659
3299	3623	4001	4357	4733	5107	5503	5861	6269	6661
3301	3631	4003	4363	4751	5113	5507	5867	6271	6673
3307	3637	4007	4373	4759	5119	5519	5869	6277	6679
3313	3643	4013	4391	4783	5147	5521	5879	6287	6689
3319	3659	4019	4397	4787	5153	5527	5881	6299	6691
3323	3671	4021	4409	4789	5167	5531	5897	6301	6701
3329	3673	4027	4421	4793	5171	5557	5903	6311	6703
3331	3677	4049	4423	4799	5179	5563	5923	6317	6709
3343	3691	4051	4441	4801	5189	5569	5927	6323	6719
3347	3697	4057	4447	4813	5197	5573	5939	6329	6733
3359	3701	4073	4451	4817	5209	5581	5953	6337	6737
3361	3709	4079	4457	4831	5227	5591	5981	6343	6761
3371	3719	4091	4463	4861	5231	5623	5987	6353	6763
3373	3727	4093	4481	4871	5233	5639	6007	6359	6779
3389	3733	4099	4483	4877	5237	5641	6011	6361	6781
3391	3739	4111	4493	4889	5261	5647	6029	6367	6791
3407	3761	4127	4507	4903	5273	5651	6037	6373	6793
3413	3767	4129	4513	4909	5279	5653	6043	6379	6803
3433	3769	4133	4517	4919	5281	5657	6047	6389	6823
3449	3779	4139	4519	4931	5297	5659	6053	6397	6827
3457	3793	4153	4523	4933	5303	5669	6067	6421	6829
3461	3797	4157	4547	4937	5309	5683	6073	6427	6833

6841	7027	7247	7499	7673	7879	8101	8297	8539	8731
6857	7039	7253	7507	7681	7883	8111	8311	8543	8737
6863	7043	7283	7517	7687	7901	8117	8317	8563	8741
6869	7057	7297	7523	7691	7907	8123	8329	8573	8747
6871	7069	7307	7529	7699	7919	8147	8353	8581	8753
6883	7079	7309	7537	7703	7927	8161	8363	8597	8761
6899	7103	7321	7541	7717	7933	8167	8369	8599	8779
6907	7109	7331	7547	7723	7937	8171	8377	8609	8783
6911	7121	7333	7549	7727	7949	8179	8387	8623	8803
6917	7127	7349	7559	7741	7951	8191	8389	8627	8807
6947	7129	7351	7561	7753	7963	8209	8419	8629	8819
6949	7151	7369	7573	7757	7993	8219	8423	8641	8821
6959	7159	7393	7577	7759	8009	8221	8429	8647	8831
6961	7177	7411	7583	7789	8011	8231	8431	8663	8837
6967	7187	7417	7589	7793	8017	8233	8443	8669	8839
6971	7193	7433	7591	7817	8039	8237	8447	8677	8849
6977	7207	7451	7603	7823	8053	8243	8461	8681	8861
6983	7211	7457	7607	7829	8059	8263	8467	8689	8863
6991	7213	7459	7621	7841	8069	8269	8501	8693	8867
6997	7219	7477	7639	7853	8081	8273	8513	8699	8887
7001	7229	7481	7643	7867	8087	8287	8521	8707	8893
7013	7237	7487	7649	7873	8089	8291	8527	8713	
7019	7243	7489	7669	7877	8093	8293	8537	8719	

APPENDIX F
COORDINATES OF DOTS ON THE TRIANGULAR ARRAY FOR 133 LINES

LINE	SPOTS ON THE LINE
1	
2	1,2
3	2
4	1
5	1,3
6	2,4
7	2
8	1,3
9	1,5,7
10	2,8
11	4,6
12	1,5,7
13	1,5,11
14	6,10,12
15	2,4,8
16	7,11
17	1,3,13,15
18	4,10,14
19	2,8,10
20	1,3,7,9
21	1,13,17,19
22	2,8,10,20
23	4,10,16,18
24	1,5,7,17
25	7,11,13,17
26	6,12,22,24
27	2,8,16,22
28	1,5,11,19,23
29	3,13,15,25,27
30	4,8,14,22,26,28
31	2,14,22,26
32	3,7,13,25,27
33	13,19,29
34	2,8,10,16,26,32
35	4,6,12,18,22,24
36	1,11,13,17,23,29,31
37	7,11,17,25,35
38	6,16,24,30,36
39	2,10,16,20,28,32
40	7,17,29,31
41	1,3,7,9,19,33,37,39
42	2,16,20,22,26
43	4,8,16,26,34,38
44	1,7,21,25,31,37
45	1,7,19,23,29,31,41,43

APPENDIX F

COORDINATES OF DOTS ON THE TRIANGULAR ARRAY FOR 133 LINES

LINE	SPOTS ON THE LINE
46	4,14,16,26,28,34
47	6,10,12,16,22,28,36,42
48	1,23,25,35,43
49	5,11,17,25,37,41,47
50	4,6,12,24,34
51	2,4,8,14,16,22,26,28,32,44,46
52	1,35,41,47
53	3,21,31,45,49,51
54	2,8,16,20,22,28,40,50,52
55	2,4,8,14,26,38,46
56	3,9,13,19,27,31,39,43
57	1,5,11,13,17,23,25,31,41
58	4,10,14,16,40,44,46,56
59	10,12,22,30,36,42,48
60	7,13,17,19,31,41,53
61	1,17,31,37,41,43,47,49,59
62	10,16,22,40,42,58,60
63	20,26,34,40,44,46,50,58
64	1,11,13,23,37,47,53
65	1,3,7,9,19,31,33,49,51,57,61,63
66	8,16,34,58,62
67	2,10,26,28,32,40,56,58,62
68	3,9,15,19,31,33,55,61,63
69	1,5,11,25,31,35,37,43,47,53,65
70	2,8,22,26,32,44,52,58,62
71	18,36,46,54,58,64,66
72	1,23,35,37,53,61,65
73	5,19,29,31,35,43,49,55,59,61,65,71
74	6,10,12,18,28,30,40,48,52,66
75	2,14,16,22,26,28,44,58,62,68
76	1,7,11,29,37,47,53,59,67
77	1,13,27,31,37,43,45,73,75
78	8,16,20,34,38,46,58,64,76
79	2,8,28,38,40,56
80	3,7,9,21,27,31,43,49,57,61,69
81	11,13,17,19,31,59,61,67,73,79
82	2,8,10,22,26,38,40,50,52,68,70
83	4,10,30,46,54,58,60,64,66
84	5,13,25,31,41,43,47,53,55,61,71,73
85	1,11,13,23,37,43,47,53,61,67,73
86	4,16,18,22,36,42,46,54,64,72,78,84
87	20,26,28,38,52,56,62,80,82
88	5,19,23,25,35,49,53,61,79,83
89	1,3,7,13,15,27,31,51,73,85,87
90	2,8,14,16,22,44,46,52,68,74,86,88

APPENDIX F
COORDINATES OF DOTS ON THE TRIANGULAR ARRAY FOR 133 LINES

LINE	SPOTS ON THE LINE
91	4,16,32,34,38,44,58,62,64,82
92	15,25,31,33,43,45,55,57,67,73,75,85,87
93	5,11,19,49,59,61,71,79,85
94	2,20,26,38,50,52,70,76,80,86,92
95	16,18,28,42,48,52,54,58,82,84
96	1,7,23,31,37,43,61,77,79,83,89,91
97	1,7,17,23,35,47,65,67,73,77,95
98	6,30,34,36,40,46,48,60,64,78
99	10,20,26,38,52,58,68,80,82,86,92
100	1,7,17,19,23,37,43,49,53,59,61,71,73,89
101	1,9,27,31,37,49,51,57,63,69,97
102	2,16,20,28,38,46,58,76,80,82,86
103	8,20,26,28,44,50,56,70,80,94,98
104	25,31,37,43,51,57,61,63,75,81,85,87,93
105	11,17,19,23,41,43,47,59,61,67,71,97,103
106	4,8,16,26,58,74,76,82,86,88,92,94,104
107	12,18,22,30,40,46,66,70,72,78
108	1,5,13,23,29,35,43,49,61,65,71,73,79,83,89,91,101,103
109	11,17,37,41,53,67,95,101
110	12,16,34,42,48,52,58,72,78,84,94,96,106
111	8,16,26,28,38,46,58,68,92,94,98,106
112	1,5,13,31,41,47,53,55,61,71,83,85,95,101,107
113	1,9,15,25,31,33,39,45,51,61,69,93,99
114	8,10,28,32,40,50,80,88,106,110,112
115	8,14,16,22,26,44,52,64,82,98,104,106
116	3,9,19,21,31,33,39,49,63,67,91,93,109,111
117	5,7,17,37,41,43,47,55,71,77,83,85,97,113
118	4,8,14,44,46,56,58,64,68,74,80,88,94,98,110,116
119	6,18,22,36,48,58,82,88,100,106,108
120	11,19,37,47,53,67,71,73,79,89,97,103,107,113
121	23,37,47,49,61,71,73,89,91,109
122	12,30,36,52,70,76,78,96,100,106,108,118
123	4,14,20,26,34,38,44,46,56,58,70,74,80,86,88,100,104,118
124	13,17,23,43,47,55,61,65,73,77,91,97,101,115
125	3,7,9,39,43,67,73,79,91,103,117,123
126	2,4,8,26,32,44,52,58,62,74,76,88,118
127	8,10,16,38,52,58,68,80,86,88,92,100,110,116,122
128	19,33,39,43,51,63,81,91,93,103,105,109,115
129	7,13,17,31,35,37,41,55,61,73,97,107,113,121
130	2,4,34,38,44,46,58,62,76,82,116,128
131	6,12,22,24,28,48,58,66,82,84,94,108,112,114,126
132	1,17,23,31,35,43,47,53,61,67,73,85,91,95,101,107,115
133	1,5,25,29,41,43,53,59,61,71,83,85,89,109,115

"LINES OF PRIMES!" SUPPLEMENT

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16. PATTERNS WITH LINE CONTINUATION

WHEN THE LINES OF PRIMES GENERATED BY THE NINE FORMULAS OF THE FAMILY OF FORMULAS REACH THE POINT WHERE THEY ARE FINALLY BROKEN, IT COULD EASILY BE ASSUMED THAT THERE IS NO MORE ORDERLINESS BEYOND THAT POINT. SUCH IS NOT THE CASE.

THERE ARE PATTERNS OF PRIMENESS AND NON-PRIMENESS FOR A LITTLE WAY BEYOND THAT BREAK FOR EACH OF THE NINE LINES.

FOR EXAMPLE, TAKE $X^2 + X + 41$. FOR THE FIRST 40 VALUES OF X (0 THROUGH 39) $X^2 + X + 41$ IS PRIME. FOR 40 AND 41 IT IS NOT PRIME. FOR 42 AND 43 IT IS. FOR 45 - 48 IT IS. FOR 50 - 55 IT IS. FOR 57 - 64 IT IS. FOR 66 - 75 IT IS.

THE PATTERN THUS FORMED LOOKS LIKE THIS:

X	X'S IN A ROW	PRIME?
0 - 39	40	Y
40 & 41	2	N
42 & 43	2	Y
44	1	N
45 - 48	4	Y
49	1	N
50 - 55	6	Y
56	1	N
57 - 64	8	Y
65	1	N
66 - 75	10	Y

HOW DOES THIS PATTERN COMPARE WITH THOSE OF THE OTHER EIGHT LINES?

HERE ARE THE PATTERNS FOR ALL NINE LINES, SIDE BY SIDE:

$X^2 + X + 5$	4 2 2
$2X^2 + 5$	5 2 1
$2X^2 + 2X + 7$	6 2 1 1 2
$X^2 + X + 11$	10 2 2 1 4
$2X^2 + 11$	11 2 1 1 2 1 3
$X^2 + X + 17$	16 2 2 1 4 1 6
$2X^2 + 2X + 19$	18 2 1 1 2 1 3 1 4
$2X^2 + 29$	29 2 1 1 2 1 3 1 4 1 5 1 6
$X^2 + X + 41$	40 2 2 1 4 1 6 1 8 1 10

FIGURE 102.

NOTICE THAT THE $X^2 + X + C$ TYPE LINES ARE THE ONES WITH THE EVEN-NUMBERED PRIME GROUPINGS AFTER THE INITIAL "BREAK". AND, OF COURSE, THE INITIAL BREAK IN EACH CASE IS SIGNALLED WITH TWO NON-PRIMES, WHILE THE SPACING AFTER THAT BETWEEN PRIME GROUPINGS, FOR THE DURATION OF THE PATTERN, IS ONE NON-PRIME. FINALLY, THESE PATTERNS EXTEND IN EACH CASE TO ROUGHLY $X = 2C$ WHERE C IS THE CON-

STANT AT THE END OF THE APPROPRIATE FORMULA.

17. LINES GALORE!

IT RECENTLY OCCURRED TO ME TO TRY NEW ARRAY VARIATIONS TO SEE IF ANY MORE LINES COULD BE FOUND. I HAD NEVER LOOKED AT THE ARRAY BASED ON COUNTING IN THIS MANNER:

```
1
2 3 4 5
6 7 8 9 10 11 12
     ETC.
```

FIGURE 103.

OR FOR THAT MATTER AT ANY ARRAYS BASED ON "JUTTING" MORE THAN TWO SPOTS TO THE RIGHT WITH EACH ARRAY LINE, KEEPING THE "JUT" CONSTANT.

WELL, THERE ARE MANY MORE LINES TO BE FOUND IN THIS WAY.

LET'S TAKE JUTTING 3 FIRST.

BY COUNTING AS IN FIGURE 103, EVIDENCE OF SEVERAL UNBROKEN LINES OF PRIMES APPEARS. SEE FIGURE 104. THERE IS ONLY ONE PROBLEM - INSTEAD OF SPANNING THE ARRAY FROM THE "HYPOTENUSE" SOUTHWEST TO THE LEFT EDGE, THESE LINES EXTEND SOUTHEAST AND STOP IN THE MIDDLE OF NOWHERE. THE ENDPOINTS ARE ALIGNED, IT IS TRUE, BUT THIS WILL NEVER DO.

IS THERE ANY WAY TO RECTIFY THIS SITUATION? ANY WAY TO GENTLY SHIFT THE LINES TO THE SOUTHWEST WITHOUT DISTURBING THEIR STRUCTURE? YES.

REMEMBERING HOW ORIGINALLY "THE OTHER FOUR LINES" WERE ORIENTED VERTICALLY WITH ENDPOINTS ALIGNED ON AN IMAGINARY LINE EXTENDING SOUTHEAST, WHEN THE COUNTING WAS AS IN FIGURE 63, WHILE THEY BEHAVED "PROPERLY" WHEN THE COUNTING WAS ALTERED SLIGHTLY AS IN FIGURE 65, I THOUGHT ABOUT TRYING SOMETHING SIMILAR.

BY CHANGING THE COUNTING SCHEME ONLY SLIGHTLY TO:

```
1  2
3  4  5  6  7
8  9 10 11 12 13 14 15
         ETC.
```

FIGURE 105.

WE GET THE DESIRED EFFECT. SEE FIGURE 106.

ALL TOLD THERE ARE NINE LINES IN THE ARRAY OF FIGURE 106, AND THEY FORM A FAMILY VERY SIMILAR TO THE ORIGINAL FAMILY OF NINE LINES. THE NINE NEW FORMULAS ARE:

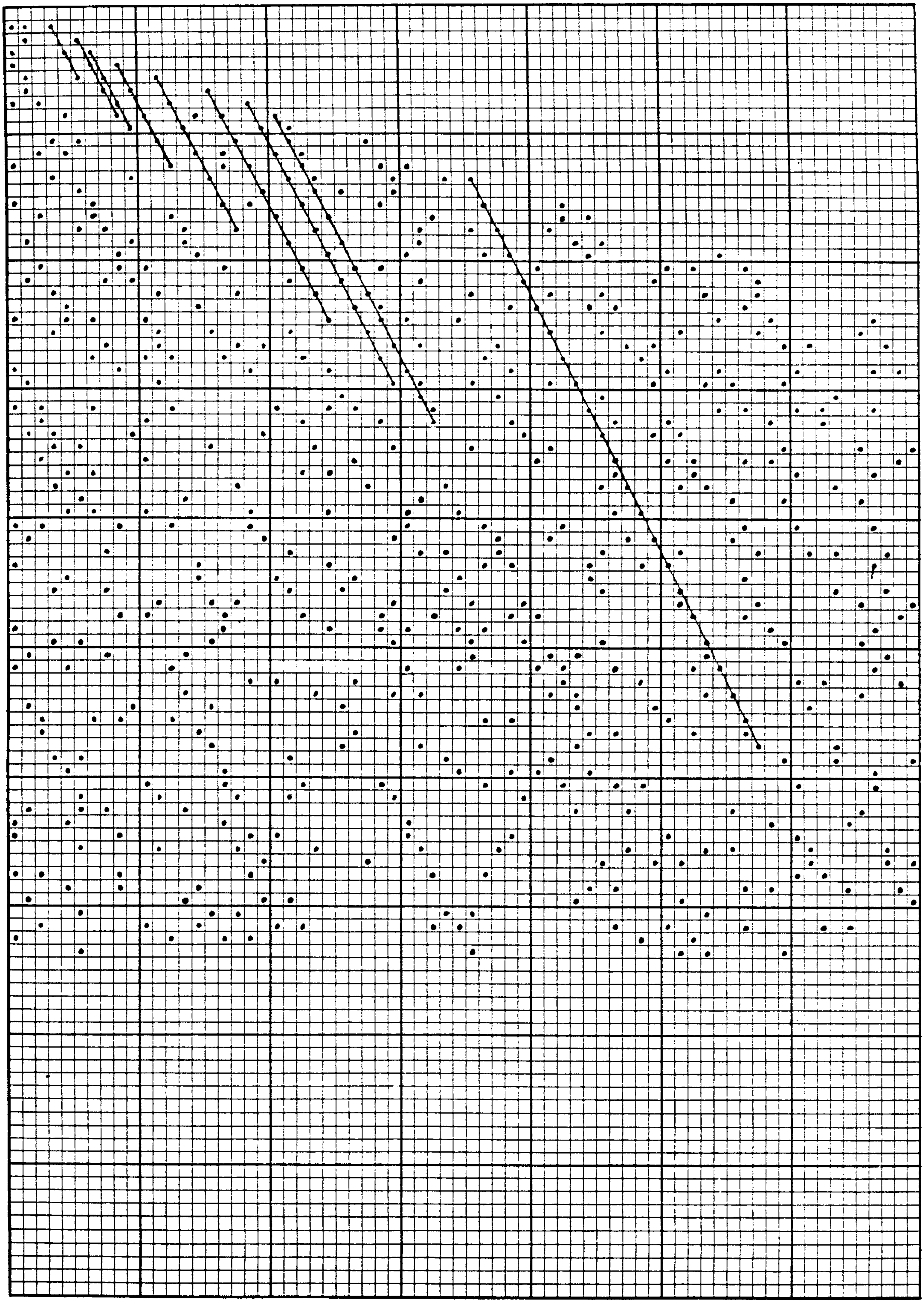

Figure 104.

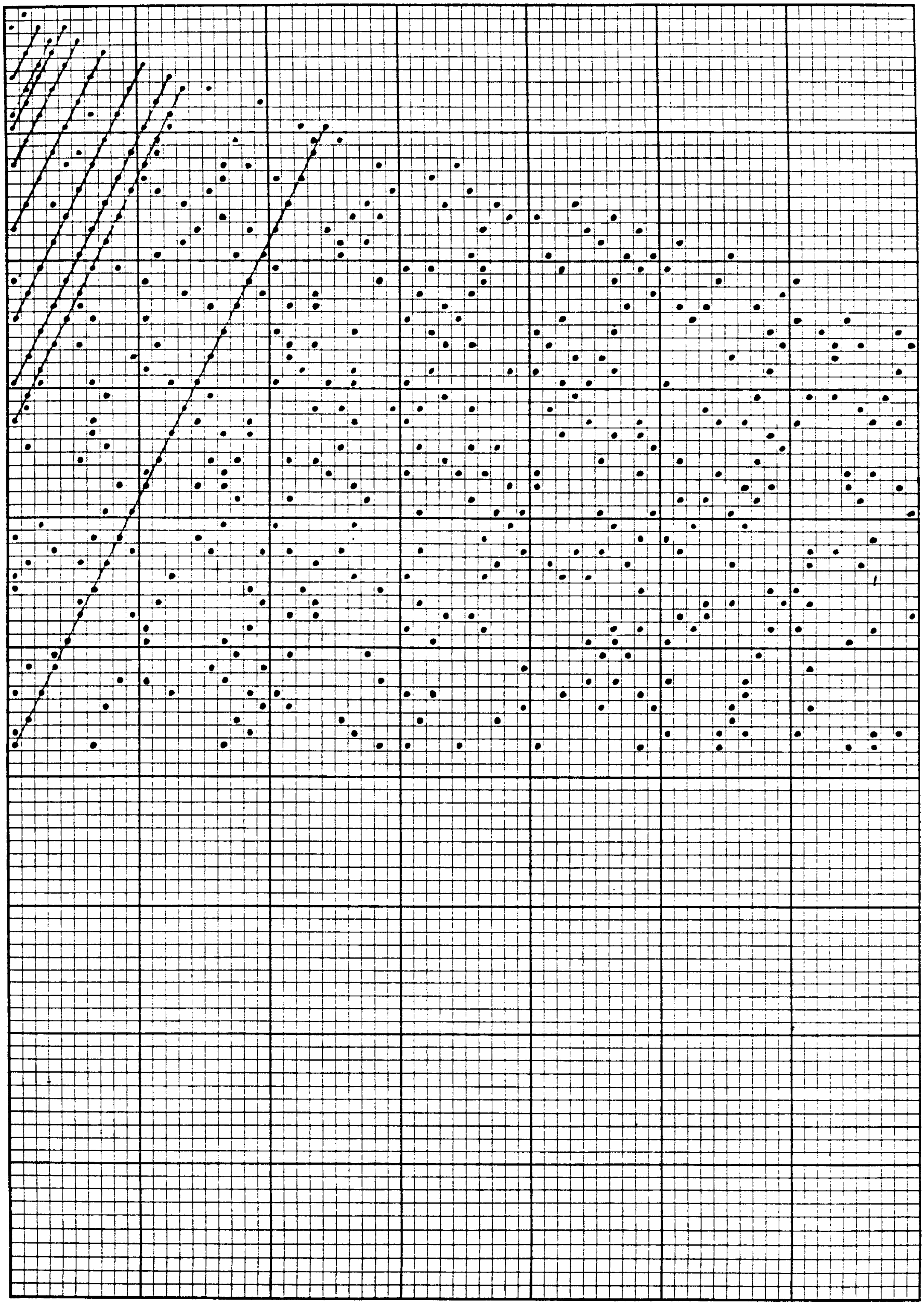

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Figure 106.

$6X^2 + 6X + 5$	$-1 < X < 3$
$6X^2 + 5$	$-1 < X < 5$
$6X^2 + 6X + 7$	$-1 < X < 5$
$6X^2 + 7$	$-1 < X < 7$
$6X^2 + 6X + 11$	$-1 < X < 9$
$6X^2 + 13$	$-1 < X < 13$
$6X^2 + 6X + 17$	$-1 < X < 15$
$6X^2 + 17$	$-1 < X < 17$
$6X^2 + 6X + 31$	$-1 < X < 29$

FIGURE 107.

AS WAS THE CASE WITH THE ORIGINAL NINE LINES, THE NORTHEAST ENDPOINTS OF THESE LINES ARE NOT THE VALUES OF THE FUNCTIONS WHEN X = 0. ALSO, AS WITH THE ORIGINAL NINE LINES, THE SOUTHWEST ENDPOINTS OF THESE NEW LINES ARE THE LAST VALUES OF THE FUNCTIONS BEFORE THE STRING OF PRIMES IS BROKEN. IN OTHER WORDS, IF $6X^2 + 6X + 11$ IS PRIME FOR NINE CONSECUTIVE VALUES OF X, FROM 0 THROUGH 8, ITS LINE ON THE ARRAY IN FIGURE 106 WILL HAVE SOMEWHAT FEWER THAN NINE DOTS IN IT, BUT THE LAST DOT GOING DOWN THE LINE WILL CORRESPOND TO THE VALUE OF $6X^2 + 6X + 11$ FOR X = 8. (THAT LAST DOT WOULD REPRESENT THE NUMBER 443.) BY THE SAME TOKEN, THE LINE REPRESENTING $6X^2 + 6X + 31$ WOULD HAVE FEWER THAN 29 DOTS BUT THE LAST DOT IN IT WOULD CORRESPOND TO THE LAST PRIME IN THE UNBROKEN STRING GENERATED BY $6X^2 + 6X + 31$, THAT OF X = 28. (THAT LAST DOT WOULD REPRESENT THE NUMBER 4903.)

WHAT ABOUT ARRAYS WITH JUTS OF GREATER THAN 3? YES, MORE LINES ARE THERE.

BY JUTTING 4 AND STARTING WITH A TOP OF ONE SPOT THUS:

```
1
2  3  4  5  6
7  8  9 10 11 12 13 14 15
             ETC.
```

FIGURE 108.

A PAIR OF LINES APPEARS, BUT THEY ARE OUR OLD FRIENDS $2X^2 + 2X + 7$ AND $2X^2 + 2X + 19$ (SEE FIGURE 109, LINES A AND B), WHILE THREE PARALLEL STRINGS HEADING SOUTHEAST WHOSE SOUTHERN ENDPOINTS LINE UP ALSO CATCH THE EYE, AND ARE OLD FRIENDS $2X^2 + 5$, $2X^2 + 11$, AND $2X^2 + 29$ (FIGURE 109, LINES C, D, AND E). ALSO, AT A DIFFERENT SLOPE THAN THE $2X^2 + 2X + C$ LINES IS A SMALL LINE OF 6 DOTS HEADING SOUTHWEST WHICH TURNS OUT TO BE ONLY ANOTHER STRETCH OF $2X^2 + 29$ (LINE F IN FIGURE 109).

FOR A JUT OF 5 AND A TOP OF 1 SPOT, ONLY ONE SHORT LINE APPEARS, TO BE DISCUSSED IN A MOMENT. IN ADDITION, A FEW PARABOLAS OF DOTS ARE EVIDENT. THESE CURVES BECOME STRAIGHT LINES WHEN THE JUT IS 6. THEIR DISCUSSION WILL ALSO BE DEFERRED (UNTIL JUT = 6 IS DISCUSSED).

THE LITTLE LINE MENTIONED A MOMENT AGO IS A BIT TROUBLESOME. IT CONFORMS IN TWO WAYS TO THE DEFINITION OF A LINE OF PRIMES, BUT

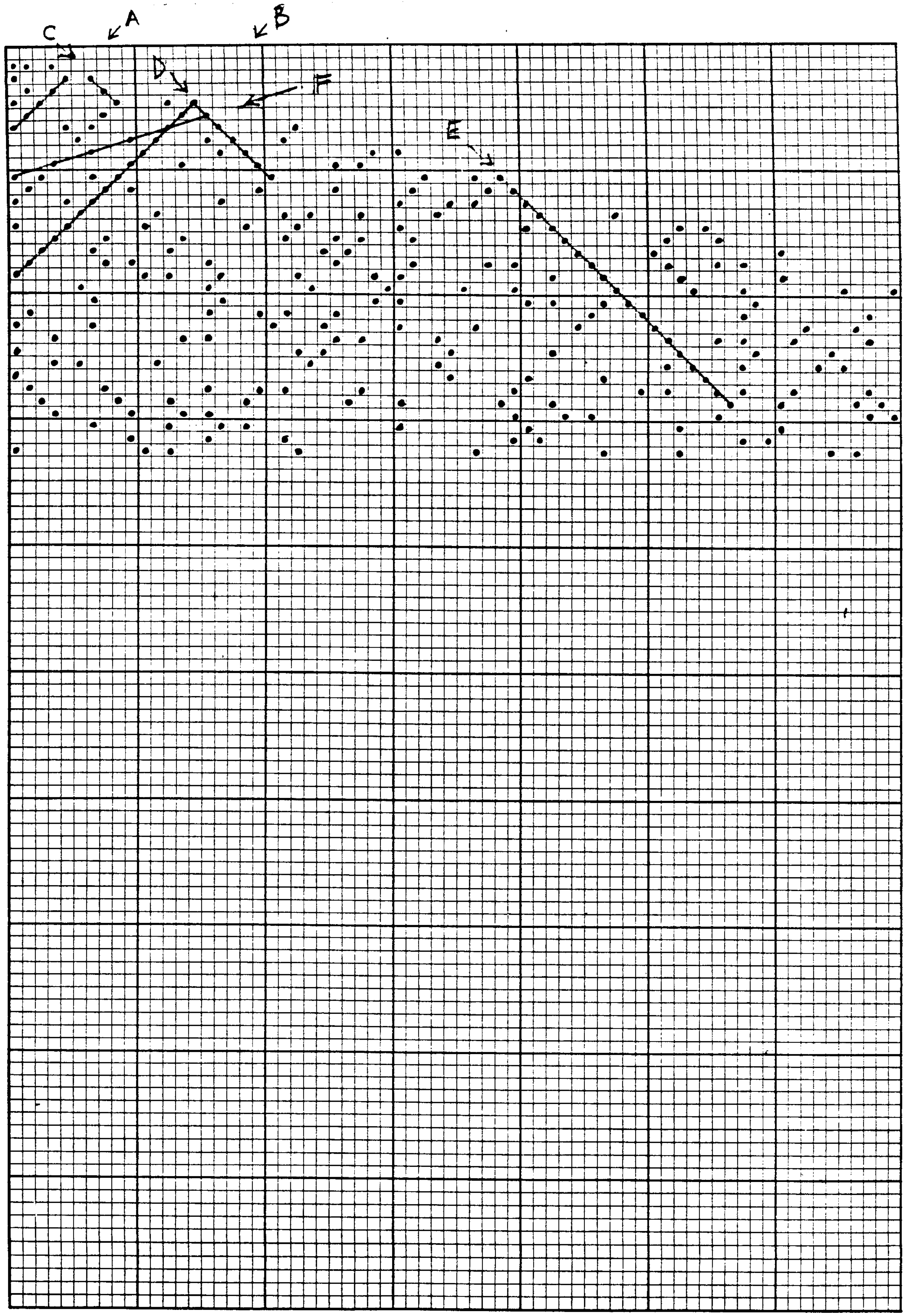

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Figure 109.

IN TWO OTHER WAYS IT DOES NOT. TO BE EXPLICIT, SO FAR PRIME LINES HAVE BEEN UNBROKEN STRAIGHT STRINGS OF EVENLY SPACED DOTS (REPRESENTING PRIME NUMBERS) AND THESE STRINGS HAVE SPANNED THE GRAPHS FROM THE "HYPOTENUSE" OF THE TRIANGULAR ARRAYS ALL THE WAY SOUTHWEST UNTIL THEY REACHED THE "WESTERN" EDGE OF THE GRAPHS. IN ADDITION, THE FORMULAS FOR THE PRIME SEQUENCES GENERATING THE LINES HAVE BEEN OF THE FORM $AX^2 + BX + C$ OR $AX^2 + C$ WHERE C IS PRIME. ALSO, THE FIRST VALUE OF X FOR WHICH THE FUNCTIONS YIELD A PRIME HAS ALWAYS BEEN ZERO AND THE LAST HAS BEEN NEAR C.

THE LITTLE LINE IN THE ARRAY WITH TOP = 1 AND JUT = 5 IS AN UNBROKEN STRING AND SPANS THE ARRAY FROM THE HYPOTENUSE SOUTHWEST TO THE WESTERN EDGE. IT DOES NOT, HOWEVER, CONFORM TO THE OTHER CRITERIA. ITS FORMULA IS $10X^2 - 4X + 5$ AND X RANGES FROM -4 THROUGH 4.

TO RECONCILE THE TWO TYPES OF PRIME LINE IS NO REAL PROBLEM, THOUGH. FIRST, A FORMULA NEED ONLY BE A QUADRATIC OF FORM $AX^2 + BX + C$. FOR THE ORIGINAL FORMULAS, EITHER B WAS EQUAL TO A OR B WAS ZERO, AND A, B, AND C WERE NON-NEGATIVE. THOSE RESTRICTIONS NEED NOT HOLD. FURTHERMORE, THE RANGE OF X WAS FROM ZERO TO SOME POSITIVE NUMBER. IT CAN NOW EXTEND FROM ANY INTEGER TO ANY OTHER, FOR EXAMPLE FROM -4 TO 4. IT TURNS OUT THAT THE RANGE OF X FOR ANY OF THE ORIGINAL FORMULAS EXTENDS JUST AS FAR BELOW ZERO AS IT DOES ABOVE ZERO. FOR EXAMPLE, TAKE $X^2 + X + 11$. THE VALUE OF THIS EXPRESSION FOR X = 1 IS 13. THE VALUE FOR X = -2 IS ALSO 13. THE VALUE FOR X = 2 IS 17 WHILE THAT FOR X = -3 IS ALSO 17, AND SO ON. SO, THE ACTUAL RANGE OF X IN THE ORIGINAL FORMULAS FOR WHICH THE FUNCTIONS YIELD A PRIME IS TWICE WHAT WAS STATED. FOR EXAMPLE, $X^2 + X + 41$ YIELDS A PRIME FOR EVERY WHOLE VALUE OF X FROM -40 THROUGH 39, OR FOR 80 STRAIGHT VALUES OF X. OF COURSE NO NEW PRIMES ARE GENERATED BY THE NEGATIVE X'S, AT LEAST FOR THE ORIGINAL FORMULAS. AS YOU WILL SEE, THERE ARE OTHER INSTANCES OF THE NEW TYPE OF FORMULA GENERATING "SPANNING DIAGONALS", FOR WHICH NEGATIVE VALUES OF X ARE NOT TRIVIAL.

LOOKING BACK, I NOW REMEMBER THAT WHEN I FIRST PLOTTED JUT = 3 TOP = 1 (WHICH LED ME EVENTUALLY TO NINE NEW STRAIGHT LINES, YOU WILL RECALL), THE FIRST THING THAT CAUGHT MY EYE WAS NOT THE BEGINNINGS OF THE STRAIGHT LINES BUT A SET OF PARABOLAS FITTING TOGETHER. THERE WERE FOUR OPENING TO THE RIGHT, WHICH TURNED OUT TO BE $2X^2 + 2X + 7$, $2X^2 + 11$, $2X^2 + 2X + 19$, AND $2X^2 + 29$, AND ONE WHICH OPENED TO THE LEFT. SEE FIGURE 110. THIS LAST ONE HAS THE FORMULA $X^2 + X - 43$ AND X HAS THE RANGE 0 THROUGH 17. 18 STRAIGHT PRIMES IS A NICE STRING. AFTER A LITTLE LOOKING I FOUND A FEW MORE RELATED STRINGS:

$X^2 + X - 73$	$-1 < X < 24$
$X^2 + X - 109$	$-1 < X < 28$
$X^2 + X - 169$	$0 < X < 25$
$2X^2 + 2X - 113$	$-1 < X < 22$

NOTE THAT IN $X^2 + X - 169$ C IS NOT PRIME.

TO WRAP UP THIS ASIDE, THE BIGGEST GENERATORS OF CONSECUTIVE

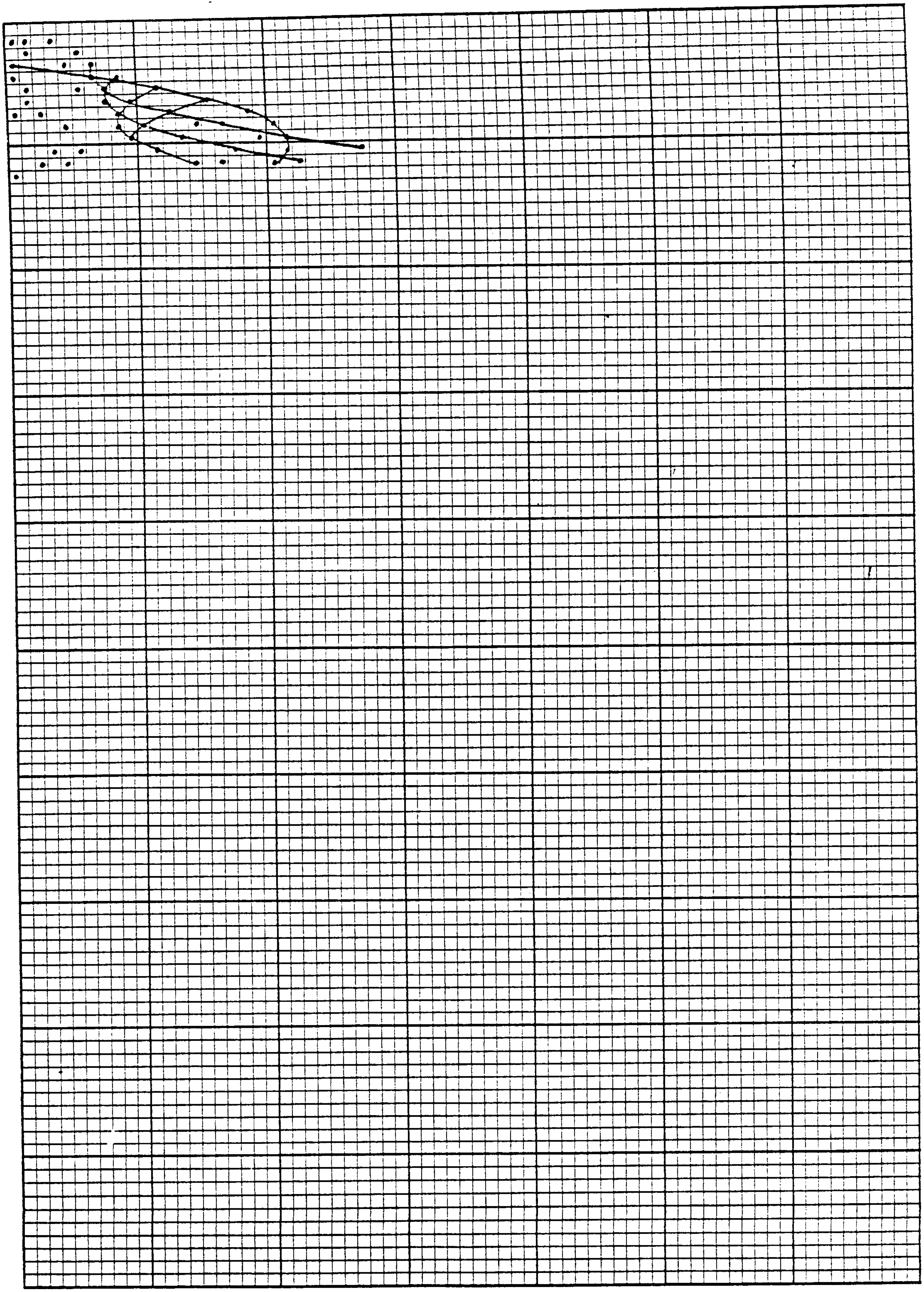

  Figure 110.

PRIMES I HAVE FOUND SO FAR BESIDE $X^2 + X + 41$ ARE QUADRATICS OF THE "NEW" TYPE, WITH B NOT = A OR 0, AND WITH X RANGING FROM NEGATIVE VALUES TO POSITIVE ONES. THE TWO LARGEST ARE:

	$25X^2 - 15X + 43$	$-8 < X < 25$
AND	$16X^2 + 28X + 53$	$-11 < X < 21$.

THE FIRST FORMULA GENERATES THE BIG PRIME RIB FEATURED IN FIGURE 62. THE SECOND WAS ALSO FOUND THROUGH THE STUDY OF PRIME RIBS.

18. JUTS OF 6 AND HIGHER

THE ARRAY BASED ON COUNTING

```
1
2  3  4  5  6  7  8
9 10 11 12 13 14 15 16 17 18 19 20 21
                 ETC.
```

FIGURE 111.

REVEALS NEW LINES:

$3X^2 + 3X + 5$	$-1 < X < 4$
$3X^2 + 3X + 11$	$-1 < X < 10$
$3X^2 + 3X + 23$	$-1 < X < 22$.

FIGURE 112.

SEE FIGURE 113.

THESE WERE THE PARABOLAS OF JUT=5.

FOR JUT=7 AND TOP=1, ONE STRAIGHT LINE APPEARS: $14X^2 + 8X + 7$, $-7 < X < 6$. IN ADDITION, ONE RATHER FERTILE PARABOLA, $4X^2 + 2X + 17$, WHICH YIELDS 16 STRAIGHT PRIMES FOR $-9 < X < 8$, APPEARS.

BEFORE WE RESUME THE DISCUSSION OF LARGER JUTS, ANOTHER SIDE-TRACK MUST BE EXPLORED, AS YOU WILL SEE:

19. WELL I'LL BE! (OLD FRIENDS IN DISGUISES)

WHEN I LOOKED AT THE ARRAY FOR JUT=8 TOP=1 TO DISCUSS THAT NEXT, JUST NOW, BESIDES SEEING THE ONE PALTRY SPANNING DIAGONAL THERE, $4X^2 + 4X + 5$, ONLY FOUR DOTS IN SPAN (SEE FIGURE 114, LINE A), I NOTICED A FEW LINES OF DOTS EXTENDING PARALLEL TO EACH OTHER IN A SOUTHEAST DIRECTION (FIGURE 114, LINES B, C, AND D). LOOKING CLOSER I DETERMINED THAT THEIR ENDPOINTS DO LINE UP, A SIGN OF MORE THAN RANDOMNESS, AN INDICATION THAT WITH THE PROPER ADJUSTMENT OF THE COUNTING BEHIND THE ARRAY THESE LINES COULD FORM A FAMILY OF SPANNING DIAGONALS FROM THE HYPOTENUSE TO THE WEST EDGE.

JOTTING DOWN THE NUMBERS THAT THESE DOTS REPRESENT, I NOTICED THAT THE BIGGEST LINE, OF 5 DOTS (LINE D), SHARES THOSE 5 VALUES, IN

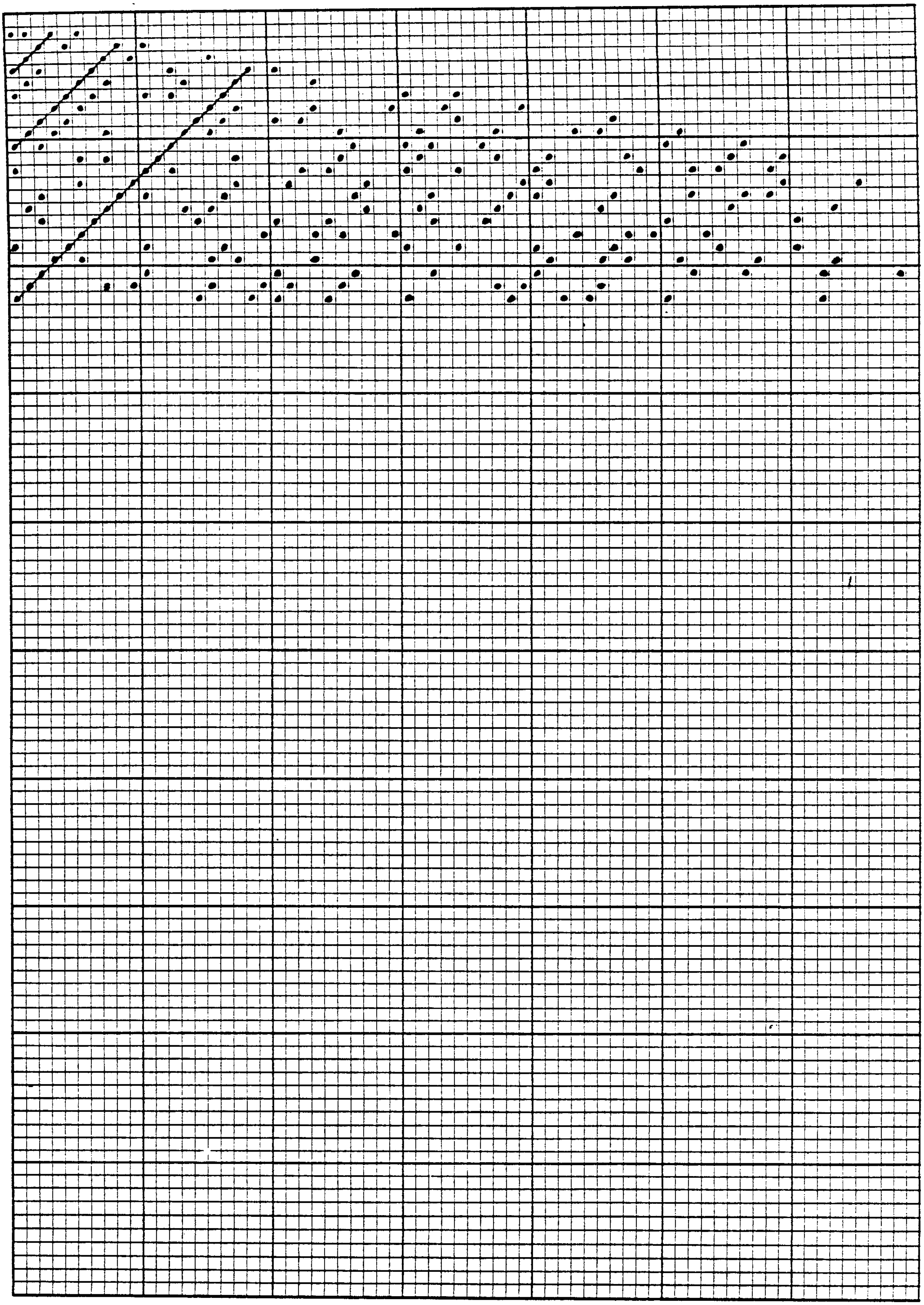

Figure 113.

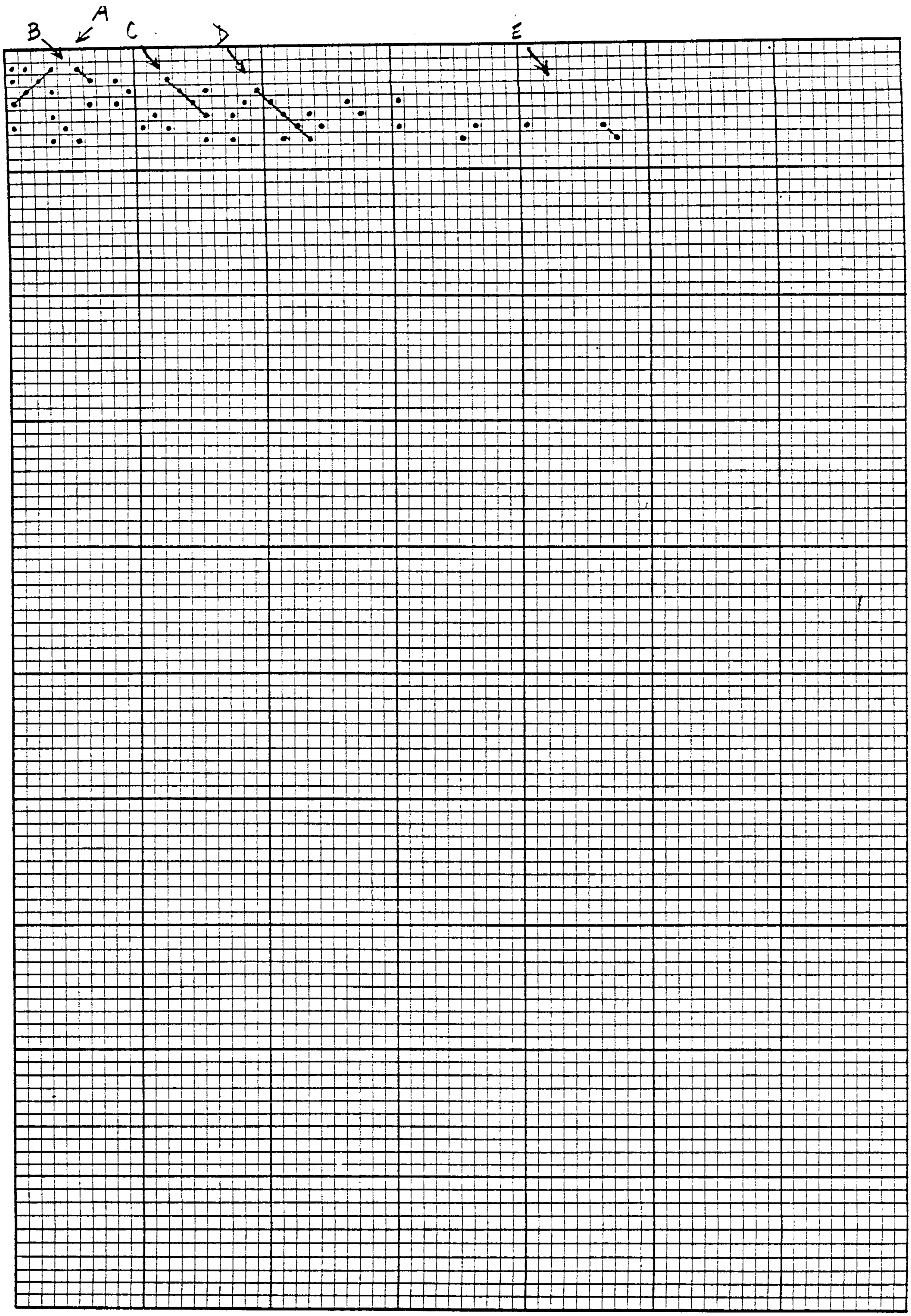

Figure 114.

ORDER, WITH THE LIST OF VALUES FOR $4X^2 + 2X + 17$, THE PARABOLA I HAD JUST INTRODUCED IN THE DISCUSSION OF JUT=7. THE FIVE DOTS HERE, HOWEVER, CORRESPONDED TO FIVE CONSECUTIVE VALUES OF $4X^2 + 2X + 17$ WHERE X IS NEGATIVE. SHRUGGING THIS OFF AS INDICATING MY LINE OF 5 DOTS IS A STRAIGHT VARIATION OF $4X^2 + 2X + 17$ ON THE JUT=8 ARRAY, I DECIDED TO LOOK FOR ANY LARGER LINES PARALLEL TO THE 5-DOT LINE. SINCE MY GRAPH WASN'T VERY EXTENSIVE, THERE WASN'T MUCH OUT TO THE RIGHT TO WORK WITH, JUST A PAIR OF DOTS WAY ON THE RIGHT THAT COULD BE THE START OF A LINE GOING SOUTHEAST (FIGURE 114, LINE E). JOTTING THEIR NUMERICAL VALUES DOWN, I STARTED EXTRAPOLATING THEM FORWARD AND BACKWARD (BASED ON THE RATE OF GROWTH IN THE FIRST SMALL PARALLEL LINES, INCLUDING THE ONE WITH 5 DOTS) TO SEE HOW MANY PRIMES THEY WOULD LEAD TO, AND THUS IF THEY WERE PART OF A SIMILAR PARALLEL LINE.

ALL TOLD, 40 STRAIGHT VALUES OF X GAVE PRIME RESULTS, FROM -19 THROUGH 20, AND THE FORMULA IS $4X^2 - 2X + 41$.

BEFORE I EVEN GOT TO DERIVE THE FORMULA, THOUGH, I FIRST WAS EXCITED TO FIND SUCH A BIG GENERATOR BUT THEN ALMOST IMMEDIATELY NOTICED SOMETHING FISHY -- THE HIGHEST PRIME GENERATED IS 1601, ALSO THE HIGHEST PRIME GENERATED BY THE REIGNING CHAMP, $X^2 + X + 41$. LOOKING CLOSER NOW AT THE NUMBERS IN $4X^2 - 2X + 41$ IT HIT ME ALL AT ONCE -- THEY ARE THE SAME EXACT NUMBERS AS THOSE GIVEN BY $X^2 + X + 41$. ONLY THE PRESENTATION VARIES.

WHILE $X^2 + X + 41$ GIVES THE 40 PRIMES IN ONE SEQUENCE STARTING WITH 41, 43, 47, 53, 61, 71, 83, ETC., AS X CLIMBS FROM ZERO ON UP TO 39, $4X^2 - 2X + 41$ GIVES THEM IN TWO SEQUENCES. FOR VALUES OF X GREATER THAN ZERO, $4X^2 - 2X + 41$ CLIMBS FROM 43 TO 53 TO 71 AND SO ON, GIVING EVERY OTHER PRIME, WHILE NEGATIVE X'S GIVE THE OTHER HALF OF THE SET OF PRIMES. TO SHOW IT ANOTHER WAY, WITH $X^2 + X + 41$ 41 IS GIVEN BY X = 0, 43 BY X = 1, 47 BY X = 2, 53 BY X = 3, AND SO ON. FOR $X^2 - 2X + 41$, 41 IS GIVEN BY X = 0, 43 BY X = 1, 47 BY X = -1, 53 BY X = 2, 61 BY X = -2, AND SO ON, ALTERNATING BACK AND FORTH FROM POSITIVE TO NEGATIVE.

WHAT'S THE NET RESULT OF ALL THIS? WELL, IT WOULD SEEM THAT THERE CAN BE MORE THAN ONE FORMULA GENERATING THE SAME SET OF PRIMES.

WITH THIS IN MIND I TOOK ANOTHER LOOK AT THE LINE WITH 5 DOTS (FIGURE 114, LINE D). (ACTUALLY, THIS LINE IN THE CONTEXT OF JUT=8 IS $4X^2 - 2X + 17$ WHEREAS IN THE CONTEXT OF JUT=7 THE SAME SEQUENCE FORMED A PARABOLA WITH FORMULA $4X^2 + 2X + 17$.) IS $4X^2 - 2X + 17$ REALLY ANOTHER OLD FRIEND IN DISGUISE? SURE ENOUGH, IT'S $X^2 + X + 17$.

AND, OF COURSE, THE TWO SMALLER LINES PARALLEL TO $4X^2 - 2X + 17$ ARE $X^2 + X + 5$ AND $X^2 + X + 11$ ANALOGUES:

$$4X^2 - 2X + 5$$
$$4X^2 - 2X + 11$$

(FIGURE 114, LINES B AND C RESPECTIVELY).

PERHAPS, AS WE PROCEED, IT WILL TURN OUT THAT OTHER FORMULAS

WHICH GENERATE PRIMES WITH NEGATIVE X'S ALSO POSSESS COMPANION FORMULAS WHICH GENERATE THE SAME PRIMES BUT WITHOUT NEGATIVE X'S.

20. STILL LARGER JUTS

A JUT=9 TOP=1 ARRAY DOESN'T YIELD MUCH OF ANYTHING, AT LEAST DURING A PRELIMINARY PLOTTING. THERE IS A TENDENCY TOWARD SOUTHEASTERLY LINES OF DOTS MORE WIDELY SPACED THAN USUAL. ONE NOTEWORTHY INSTANCE OF THESE IS ELEVEN DOTS LONG ON THE ARRAY. ITS FORMULA IS $18X^2 - 6X + 19$ AND ALTOGETHER IT GENERATES 18 CONSECUTIVE PRIMES, FOR $-6 < X < 13$.

THE JUT-10 TOP-1 ARRAY YIELDS TWO NEW LINES:

$5X^2 + 5X + 7$	$-1 < X < 6$
$5X^2 + 5X + 13$	$-1 < X < 12$

FIGURE 115.

THE JUT-11 TOP-1 ARRAY DISPLAYS A FEW PARABOLAS, BUT THEY TURN OUT TO BE ALREADY KNOWN $6X^2 + 6X + C$ FORMULAS. INDEED, THESE SAME FORMULAS GENERATE FIVE PARALLEL STRAIGHT LINES SPANNING THE JUT-12 TOP-1 ARRAY.

SINCE ODD JUTS DIDN'T SEEM TO BE PANNING OUT, FROM THIS POINT ON I ONLY CONCENTRATED ON EVEN JUTS WHILE PLOTTING DOTS.

JUT-14 TOP-1 DIVULGES TWO MORE NEW LINES:

$7X^2 + 7X + 5$	$-1 < X < 4$
$7X^2 + 7X + 17$	$-1 < X < 16$

FIGURE 116.

JUT-16 TOP-1 HAS "HALF-DIAGONALS" THAT ONLY REACH HALFWAY TO THE WEST EDGE (SEE FIGURE 117). BY CHANGING THE TOP ROW OF THE ARRAY TO TWO SPOTS, THE HALF-LINES ARE STRETCHED THE REST OF THE WAY ACROSS (SEE FIGURE 118).

UNIQUE TO JUT-16 TOP-2 IS THAT VERTICAL COLUMNS 1 AND 2 ARE DEVOID OF DOTS, SO THE EFFECTIVE WEST EDGE IS COLUMN 3. COLUMN-1 SPOTS REPRESENT PRODUCTS OF FORM $(2N - 3)(4N - 5)$, WHERE N IS ARRAY-ROW-NUMBER, WHILE COLUMN-2 SPOTS REPRESENT EVEN NUMBERS.

BESIDES THE TWO PARALLEL SPANNING DIAGONALS (FIGURE 118, LINES A AND B):

$8X^2 + 8X + 13$	$-1 < X < 5$
$8X^2 + 8X + 31$	$-1 < X < 14$

FIGURE 119.

THERE IS ANOTHER SPANNING DIAGONAL BUT AT A STEEPER SLOPE (FIGURE 118, LINE C):

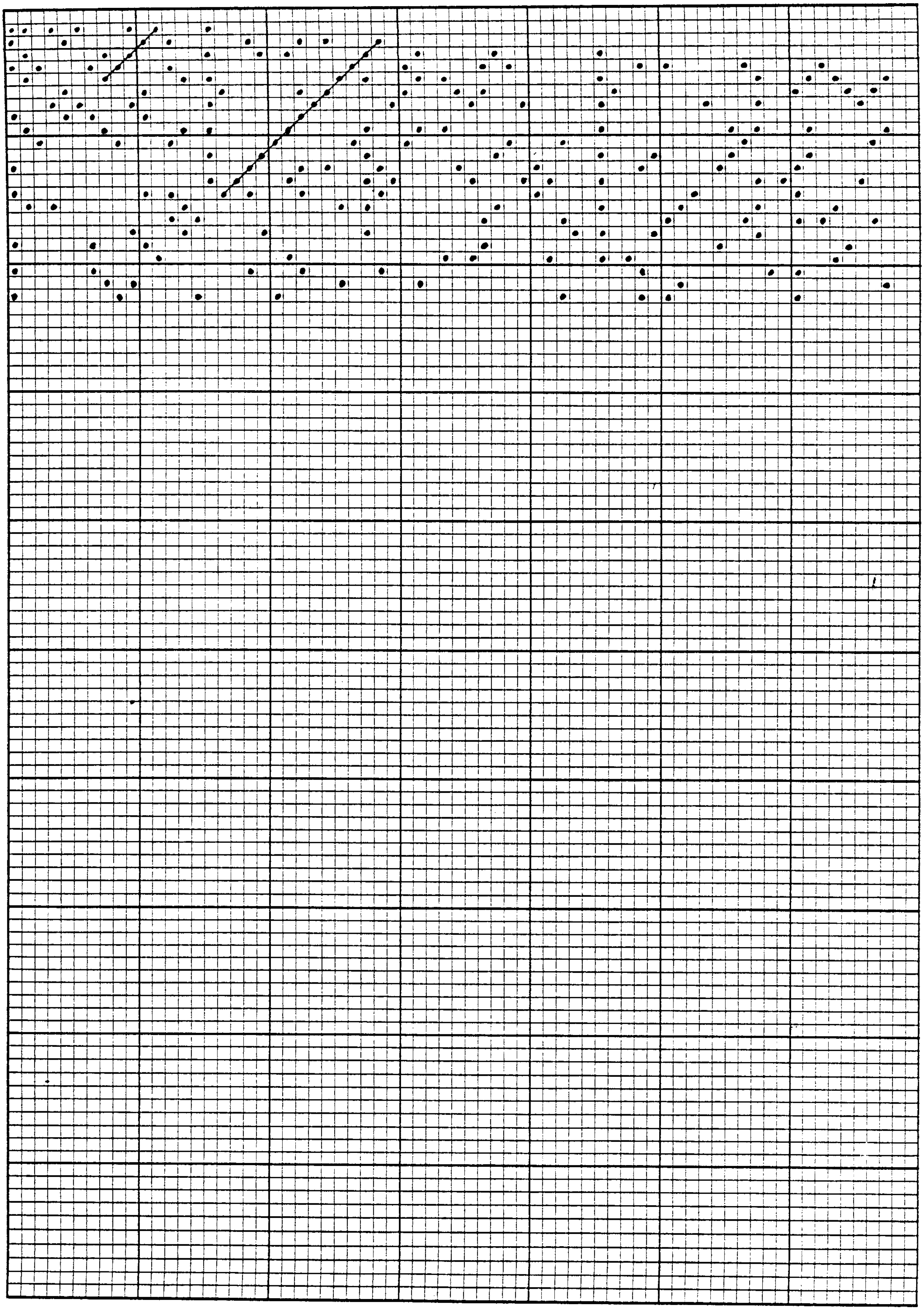

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Figure 117.

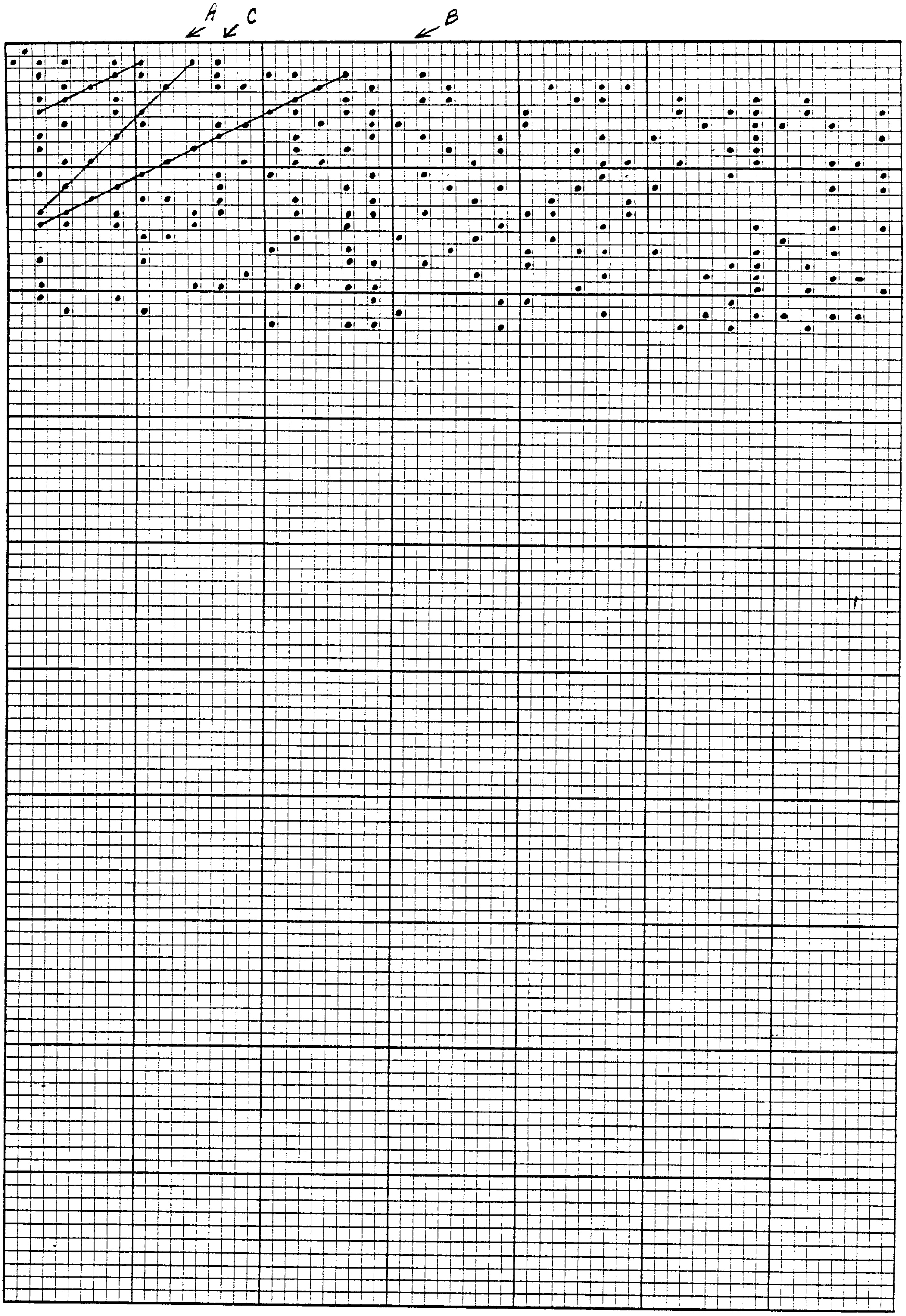

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Figure 113.

$$32X^2 - 46X + 31 \qquad -7 < X < 8.$$

AT THIS POINT I REALIZED THAT LINES OF FORMULA $AX^2 + AX + C$ WOULD APPEAR IN ARRAYS OF JUT = 2A, SO I COULD SAVE PLOTTING THE DOTS OF THE ARRAYS FROM HERE ON IN AND SIMPLY JOT DOWN NUMBER SEQUENCES OF FORM $AX^2 + AX + C$ AND CHECK THEIR MEMBERS FOR PRIMALITY. IF UNBROKEN SEQUENCES OF PRIMES TURNED UP, I COULD PREDICT WHAT ARRAY THEY WOULD BE SPANNING DIAGONALS ON -- JUT = 2A, TOP = 1, IF THE NUMBER OF PRIMES IN THE STRING EQUALS C; TOP = 2 IF THE NUMBER OF PRIMES EQUALS 1/2 (C + 1), AND SO ON FOR FRACTIONAL DIAGONALS.

FOR THAT MATTER I COULD DO THE SAME FOR SEQUENCES BASED ON FORMULAS OF THE FORM $AX^2 + C$ WHERE A IS EVEN. (ODD VALUES OF A YIELD NUMBER SEQUENCES JUMPING FROM ODD TO EVEN AND BACK, AND SO ARE OUT AS CANDIDATES FOR PRIME LINE GENERATORS.) PREDICTING THE ARRAY FOR $AX^2 + C$ LINES TURNED OUT TO BE TRICKY BUT EVENTUALLY I GOT THE HANG OF IT: JUT = 2A, TOP = A + 1, FOR "WHOLE" DIAGONALS; TOP = A + 2 FOR "HALF-DIAGONALS", AND SO ON.

BEFORE PROCEEDING ANY FURTHER I SHOULD SAY THAT WORKING STRICTLY WITH NUMBER SEQUENCES AS I DID FROM HERE ON IN HANDICAPPED FINDING NEW LINES IN TWO WAYS. FIRST, I MIGHT MISS LINES IN ARRAYS OF FORM JUT = 2A BY NOT PLOTTING THEIR DOTS; AND SECOND, I WOULD MISS ANY LINES APPEARING IN ARRAYS OF ODD JUT HIGHER THAN ELEVEN, SINCE, AS YOU WILL RECALL, I BEGAN IGNORING SUCH ODD-JUT ARRAYS. ALSO, BY LIMITING THE VALUES OF C TO SMALL PRIMES, USUALLY ONLY UP TO 47, I WAS SHUTTING OUT THE VAST MAJORITY OF POSSIBLE C'S. (IF THE TASK WERE AUTOMATED, MUCH BROADER RANGES OF SEARCH WOULD BECOME FEASIBLE.)

21. ALL DONE WITH NUMBERS

THE NEXT VALUE OF A TESTED FOR $AX^2 + AX + C$ WAS 9, BUT BEFORE DISCUSSING IT I WILL BRIEFLY EXPLAIN WHAT OTHER LINES TURNED UP BY THIS NUMERICAL ANALYSIS WHEN APPLIED RETROSPECTIVELY TO A'S SMALLER THAN 9 FOR $AX^2 + C$.

AN ORDINARILY TRIVIAL LINE CONSISTING OF ONLY THREE DOTS GENERATED BY $2X^2 + 3$ NEVERTHELESS SPANS ARRAY JUT-4 TOP-3. THIS FORMULA FITS IN THE FAMILY OF $2X^2 + C$ FORMULAS INCLUDING $2X^2 + 5$, $2X^2 + 11$, AND $2X^2 + 29$.

TWO FORMULAS IN THE $4X^2 + C$ FAMILY TURN UP:

$$4X^2 + 3 \qquad -1 < X < 3$$
$$4X^2 + 7 \qquad -1 < X < 7.$$

TOGETHER WITH THE KNOWN $4X^2 + 4X + C$ TYPE FORMULA, $4X^2 + 4X + 5$, THESE FORM A MINI-FAMILY:

$$4X^2 \qquad + 3$$
$$4X^2 + 4X + 5$$

$4X^2 \quad + 7.$

LOOKING INTO $8X^2 + C$, THREE "HALF-DIAGONALS" TURN UP:

$8X^2 + 5$	$-1 < X < 3$
$8X^2 + 11$	$-1 < X < 6$
$8X^2 + 29$	$-1 < X < 15.$

TAKING INTO ACCOUNT THE TWO $8X^2 + 8X + C$ FORMULAS DISCUSSED EARLIER AND ARRANGING THESE FIVE FORMULAS IN ORDER, THEN, BY THE NUMBER OF CONSECUTIVE VALUES OF X FOR WHICH THEY GENERATE PRIMES, THEY FORM THE FOLLOWING LIST:

FORMULA	PRIMES IN A ROW
$8X^2 + 5$	3
$8X^2 + 8X + 13$	5
$8X^2 + 11$	6
$8X^2 + 8X + 31$	14
$8X^2 + 29$	15

FIGURE 120.

RETURNING TO $9X^2 + 9X + C$, NO LINES SHOW UP. THIS NEGATIVE "CROP" UNFORTUNATELY INCREASINGLY BECOMES THE RULE AS A INCREASES. JUST AS DISAPPOINTING IS THE TENDENCY FOR THOSE LINES THAT DO E-MERGE TO BE OF MINIMAL LENGTH, BEING HARDLY EVER GREATER THAN 11 NUMBERS LONG. WITH THAT IN MIND, LET'S LOOK AT $10X^2 + C$, THE LAST REALLY RICH FIND.

$10X^2 + C$ PRODUCES FOUR PARALLEL STRAIGHT UNBROKEN ARRAY-SPANNING DIAGONAL LINES:

$10X^2 + 3$	$-1 < X < 3$
$10X^2 + 7$	$-1 < X < 7$
$10X^2 + 13$	$-1 < X < 13$
$10X^2 + 19$	$-1 < X < 19.$

THE ARRAY THAT THESE SHOW UP ON IS JUT-20 TOP-11 (SEE FIGURE 121). (NOTE: FIGURE 121 ONLY DISPLAYS THE LEFTMOST 53 COLUMNS OF THE ARRAY -- SINCE EACH ROW OF THE ARRAY IS 20 SPOTS LONGER THAN ITS PREDECESSOR, MOST OF THE ARRAY IS TOO WIDE TO FIT HERE. ALSO, ONLY THE TOP 19 ROWS OF THE ARRAY ARE PORTRAYED IN FIGURE 121.)

HERE ARE THE REST OF THE LINES THAT I HAVE FOUND UP TO A = 100:

FORMULA	PRIMES IN A ROW	ARRAY JUT	ARRAY TOP
$11X^2 + 11X + 7$	6	22	1
$12X^2 + 19$	5	24	16
$12X^2 + 31$	8	24	16
$12X^2 + 59$	15	24	16
$13X^2 + 13X + 5$	4	26	1
$13X^2 + 13X + 11$	10	26	1

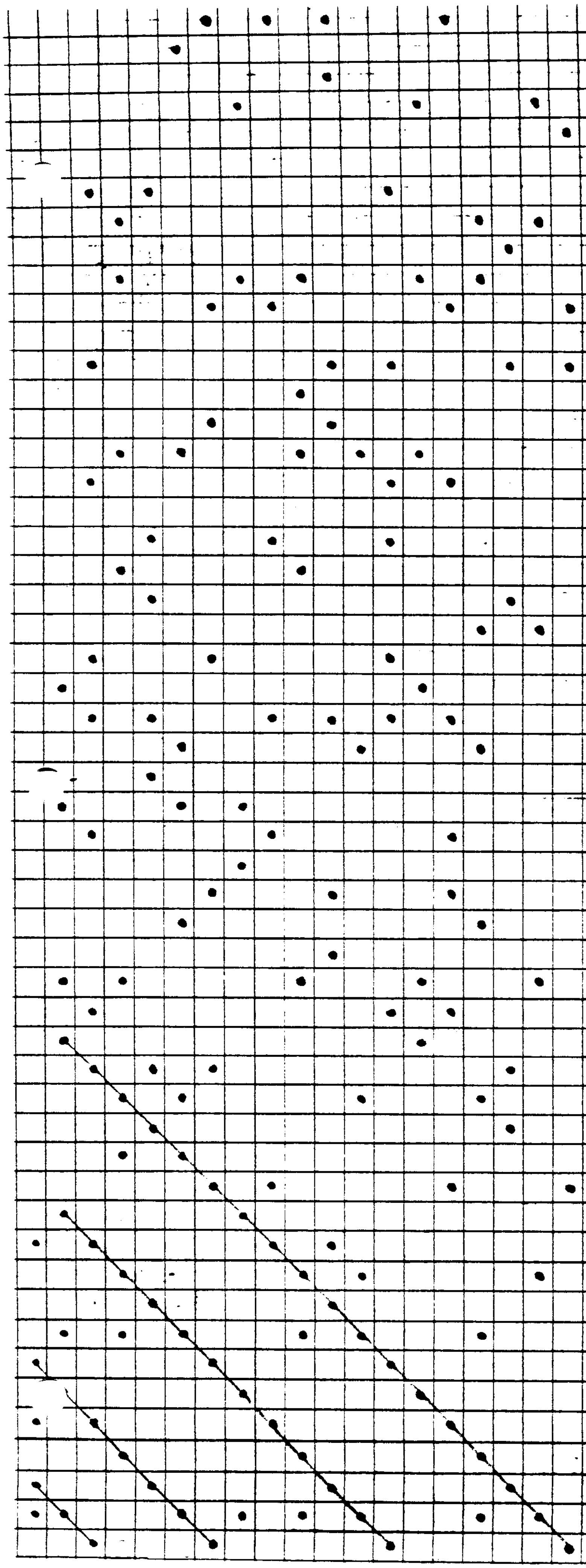

Figure 121.

$14X^2 + 3$	3	28	15
$14X^2 + 5$	5	28	15
$14X^2 + 14X + 13$	12*	28	1
$15X^2 + 15X + 7$	6*	30	1
$15X^2 + 15X + 11$	10*	30	1
$15X^2 + 15X + 13$	12*	30	1
$15X^2 + 15X + 17$	16*	30	1
$18X^2 + 5$	2	36	21
$18X^2 + 11$	4	36	21
$18X^2 + 29$	10	36	21
$20X^2 + 20X + 19$	3**	40	4
$20X^2 + 20X + 31$	6**	40	4
$20X^2 + 20X + 43$	9**	40	4
$22X^2 + 22X + 17$	16*	44	1
$24X^2 + 5$	3	48	26
$24X^2 + 24X + 13$	3**	48	2
$24X^2 + 7$	4	48	26
$24X^2 + 24X + 19$	6**	48	2
$24X^2 + 13$	7	48	26
$24X^2 + 24X + 23$	8**	48	2
$24X^2 + 17$	9	48	26
$25X^2 + 25X + 47$	16***	50	3
$30X^2 + 30X + 11$	3**	60	1
$30X^2 + 30X + 13$	5**	60	1
$30X^2 + 30X + 19$	11**	60	1
$30X^2 + 7$	7	60	31
$30X^2 + 11$	11	60	31
$35X^2 + 35X + 19$	18*	70	1
$36X^2 + 36X + 7$	6	72	1
$40X^2 + 3$	3	80	42
$40X^2 + 40X + 17$	3**	80	2
$40X^2 + 7$	5	80	42
$40X^2 + 40X + 23$	6**	80	2
$40X^2 + 13$	8	80	42
$40X^2 + 40X + 29$	9**	80	2
$40X^2 + 19$	11	80	42
$42X^2 + 5$	5	84	43
$42X^2 + 11$	11	84	43
$66X^2 + 5$	5	132	67
$66X^2 + 7$	7	132	67
$94X^2 + 7$	7	188	95

FIGURE 122.

* INTERRUPTING COLUMN PRESENT
** BARRIER COLUMN (EFFECTIVE WEST EDGE) PRESENT
*** LAST DOT PAST WEST EDGE BY 2 SPOTS

NOTE: FORMULAS GENERATING ONLY 5 OR FEWER STRAIGHT PRIMES ARE COUNTED ONLY IF THEY FORM PART OF A SET OF RELATED GENERATORS WITH THE COEFFICIENT A. GENERATORS WITH A GIVEN COEFFICIENT ARE RELATED IF A PATTERN IS FORMED BY EACH GENERATOR'S VALUE OF C VERSUS THE NUMBER OF PRIMES IN A ROW THEY EACH PRODUCE. SEE FOR EXAMPLE THE THREE $12X^2 + C$ FORMULAS. THERE THE NUMBER OF PRIMES IN A ROW FOR EACH FORMULA EQUALS (C + 1) / 4.

ALSO, PRIME SEQUENCES LONGER THAN 5 HAVE BEEN OMITTED WHEN THEY BEAR NO OBVIOUS RELATIONSHIP TO THEIR C.

22. LINE SPECIES

THERE ARE SEVERAL SPECIES OF PRIME LINES THAT HAVE TURNED UP SO FAR. THIS DISCUSSION WILL EXPLAIN HOW A LOT OF THE LINES IN THE PRECEDING EXTENSIVE LIST COME TO BE MEMBERS IN THE LIST. ALONG THE WAY, THE UNFAMILIAR TERMS USED IN THE FOOTNOTES TO THE LIST WILL BEGIN TO TAKE ON MEANING.

IN GENERAL, INCLUSION IN THE LIST OF PRIME LINES ASSUMES A TRIANGULAR COUNTING ARRAY AS THE CONTEXT. SUCH AN ARRAY HAS A TOP ROW OF SOME NUMBER OF SPOTS AND EACH SUBSEQUENT ROW JUTS OUT TO THE RIGHT BY SOME CONSTANT QUANTITY OF SPOTS. THE LEFT ENDS OF ALL ROWS ARE VERTICALLY ALIGNED -- IN OTHER WORDS, THERE IS NO JUTTING ON THE LEFT. COUNTING IS OF COURSE LEFT TO RIGHT. A SPOT REPRESENTING A PRIME NUMBER IS MARKED WITH A DOT, WHILE ALL OTHER SPOTS ARE LEFT VACANT.

GIVEN SUCH A TRAINGULAR ARRAY, A PRIME LINE IS A DIAGONAL LINE OF DOTS ON THE ARRAY SLANTING FROM NORTHEAST TO SOUTHWEST AND SPANNING ACROSS THE ARRAY FROM THE UPPER RIGHT EDGE, OR "HYPOTENUSE", TO THE WEST EDGE. ONE FURTHER RESTRICTION ON THE DOTS OF A PRIME LINE IS THAT THEY BE EVENLY SPACED ALONG THE DIAGONAL.

INHERENT IN PRIME LINES IS THAT THEY REPRESENT UNBROKEN SEQUENCES OF PRIME NUMBERS. THE SEQUENCES ARE UNBROKEN IN THE SENSE THAT FOR CONSECUTIVE WHOLE VALUES OF X PLUGGED INTO AN EXPRESSION, THE EXPRESSION PRODUCES THESE PRIME NUMBERS. USUALLY MOST BUT NOT ALL OF THE CONSECUTIVELY PRODUCED PRIMES ARE REPRESENTED IN THE STRING OF DOTTED SPOTS IN A SPANNING DIAGONAL. OFTEN THE FIRST PRIMES IN THE UNBROKEN PROGRESSION ARE NOT IN THE DIAGONAL, WHILE THE DIAGONAL CONTAINS THE REST OF THEM, THE LAST DOT ON THE DIAGONAL COINCIDING WITH THE LAST PRIME IN THE UNBROKEN PROGRESSION.

THE FIRST SPECIES OF PRIME LINE IS THE IDEAL PRIME LINE. IT HAS ALL OF THE ABOVE CHARACTERISTICS. IN ADDITION, THE LAST DOT IN THE DIAGONAL OF AN IDEAL PRIME LINE IS IN COLUMN 1, THE LEFTMOST COLUMN OF THE ARRAY. FOR AN IDEAL PRIME LINE THAT DOT ALSO REPRESENTS THE LAST PRIME IN THE UNBROKEN PROGRESSION OF PRIMES GENERATING THE LINE. AN EXAMPLE OF AN IDEAL PRIME LINE IS GENERATED BY THE EXPRESSION $X^2 + X + 41$ ON ARRAY JUT-2 TOP-1.

THE SECOND SPECIES OF PRIME LINE IS THE INTERRUPTED PRIME LINE. SUCH A DIAGONAL HAS ONE NON-PRIME SPOT IN ITS OTHERWISE UNBROKEN STRING OF DOTS. AN INTERESTING PHENOMENON INVOLVING INTERRUPTED PRIME LINES OCCURS WHEN THERE IS MORE THAN ONE PRIME LINE ON AN ARRAY. IF ANY OF THE LINES ON THE ARRAY IS INTERRUPTED, ALL WILL BE. BUT NOT ONLY THAT, ALL OF THE LINES WILL BE INTERRUPTED IN THE SAME COLUMN. WHATEVER COLUMN THAT HAPPENS TO BE, IT CONTAINS NO PRIMES WHATSOEVER. SEE THE FOUR INTERRUPTED DIAGONALS IN FIGURE 123, INTERRUPTED BY COLUMN 3. THESE FOUR PRIME LINES ARE GENERATED BY THE FOUR EXPRESSIONS:

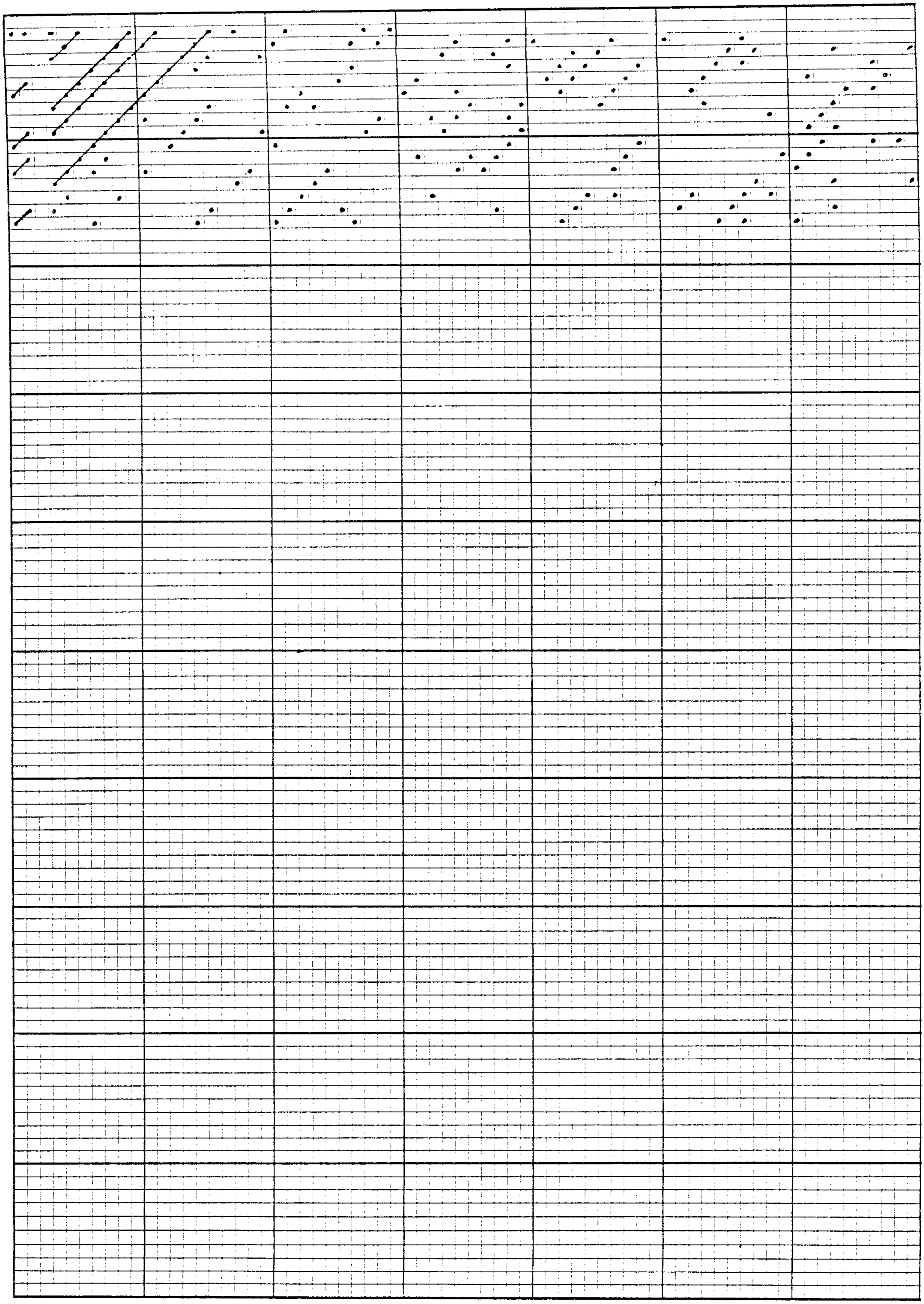

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Figure 123.

$$15X^2 + 15X + 7$$
$$15X^2 + 15X + 11$$
$$15X^2 + 15X + 13$$
$$15X^2 + 15X + 17.$$

THE THIRD SPECIES OF PRIME LINE IS THE TRUNCATED PRIME LINE. THESE DIAGONALS ARE NOT ONLY INTERRUPTED BY A COLUMN BUT ARE HALTED BY IT AS WELL. AGAIN, WHEN MORE THAN ONE PRIME LINE IS PRESENT IN AN ARRAY, IF ANY DIAGONAL MEETS SUCH A BARRIER COLUMN, ALL DIAGONALS ARE TRUNCATED BY THAT COLUMN. EXAMPLES ARE

$$30X^2 + 30X + 11$$
$$30X^2 + 30X + 13$$
AND $$30X^2 + 30X + 19$$

ON THE ARRAY WITH JUT-60 AND TOP-1. THE BARRIER COLUMN IS COLUMN 7. (SEE FIGURE 124.)

THE FOURTH SPECIES OF PRIME LINE IS THE STRETCHED PRIME LINE. STRETCHED PRIME LINES ARE FORMED FROM FRACTIONAL DIAGONALS BY STRETCHING THEM THE REST OF THE WAY ACROSS THE ARRAY TO THE WEST EDGE. THE STRETCHING IS ACCOMPLISHED BY ADJUSTING THE LENGTH OF THE TOP ROW OF THE ARRAY. GIVEN THAT IDEALLY FOR $AX^2 + AX + C$ TYPE LINES THE TOP HAS ONE SPOT AND FOR $AX^2 + C$ TYPE LINES THE TOP HAS A + 1 SPOTS, TO STRETCH "HALF-DIAGONALS" WE MAKE THE TOP ROW 2 SPOTS OR A + 2 SPOTS RESPECTIVELY, FOR "THIRD-DIAGONALS" WE MAKE THE TOP 3 SPOTS OR A + 3 SPOTS RESPECTIVELY, FOR "QUARTER-DIAGONALS" WE MAKE THE TOP 4 SPOTS OR A + 4 SPOTS RESPECTIVELY, AND SO ON. FOR AN ARRAY WITH MORE THAN ONE PARALLEL FRACTIONAL DIAGONAL, STRETCHING AFFECTS EACH SUCH DIAGONAL IN THE SAME WAY. FOR EXAMPLE, THE THREE "QUARTER-DIAGONALS" ON ARRAY JUT-24 TOP-13 GENERATED BY

$$12X^2 + 19$$
$$12X^2 + 31$$
AND $$12X^2 + 59$$

ARE STRETCHED THE REST OF THE WAY TO THE WEST EDGE BY MAKING THE TOP ROW 16 SPOTS LONG (SEE FIGURES 125 AND 126).

THE FIFTH SPECIES OF PRIME LINE IS THE OVERSTRETCHED PRIME LINE. THESE ARE STRETCHED PRIME LINES WHICH NOT ONLY SPAN TO THE WEST EDGE BUT WOULD EXTEND ONE DOT PAST THE WEST EDGE IF SUCH EXTENSION WERE PERMITTED. FOR AN OVERSTRETCHED PRIME LINE, THE NEXT-TO-LAST NUMBER IN THE UNBROKEN SEQUENCE OF PRIMES ENDS UP BEING REPRESENTED BY THE LAST DOT ON THE SPANNING DIAGONAL. AN EXAMPLE OF AN OVERSTRETCHED PRIME LINE IS GENERATED BY $25X^2 + 25X + 47$ ON ARRAY JUT-50 TOP-3 (SEE FIGURE 127).

FINALLY, THE SIXTH SPECIES OF PRIME LINE SO FAR ENCOUNTERED IS THE TRUNCATED STRETCHED OR UNDERSTRETCHED PRIME LINE. SUCH A LINE HAS BEEN STRETCHED AS FAR AS POSSIBLE TOWARD THE WEST EDGE BUT WITHOUT GOING "PAST" IT, AND HAS COME UP AGAINST A BARRIER COLUMN. WHEN A SET OF PARALLEL FRACTIONAL PRIME LINES ON AN ARRAY IS

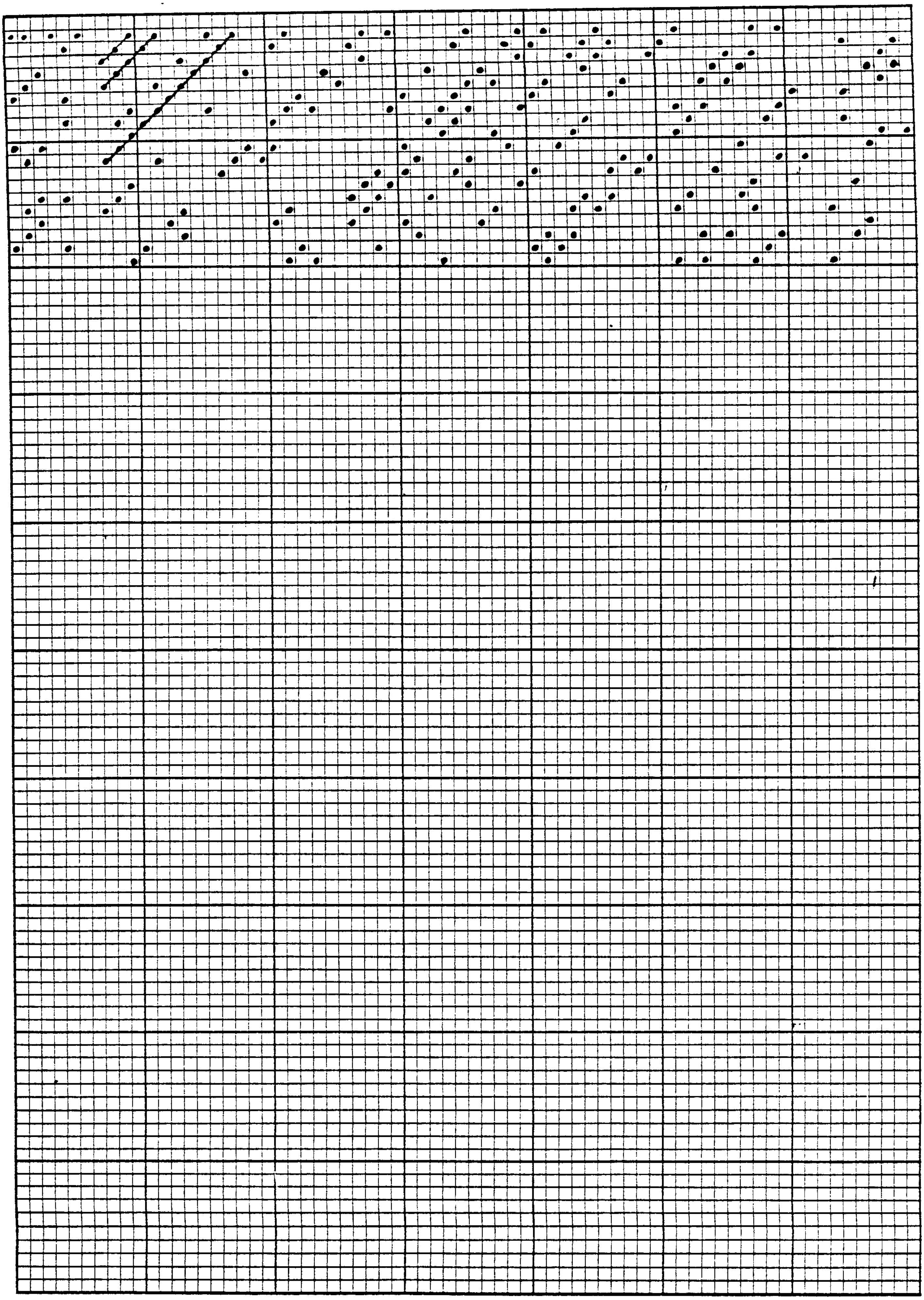

Figure 124.

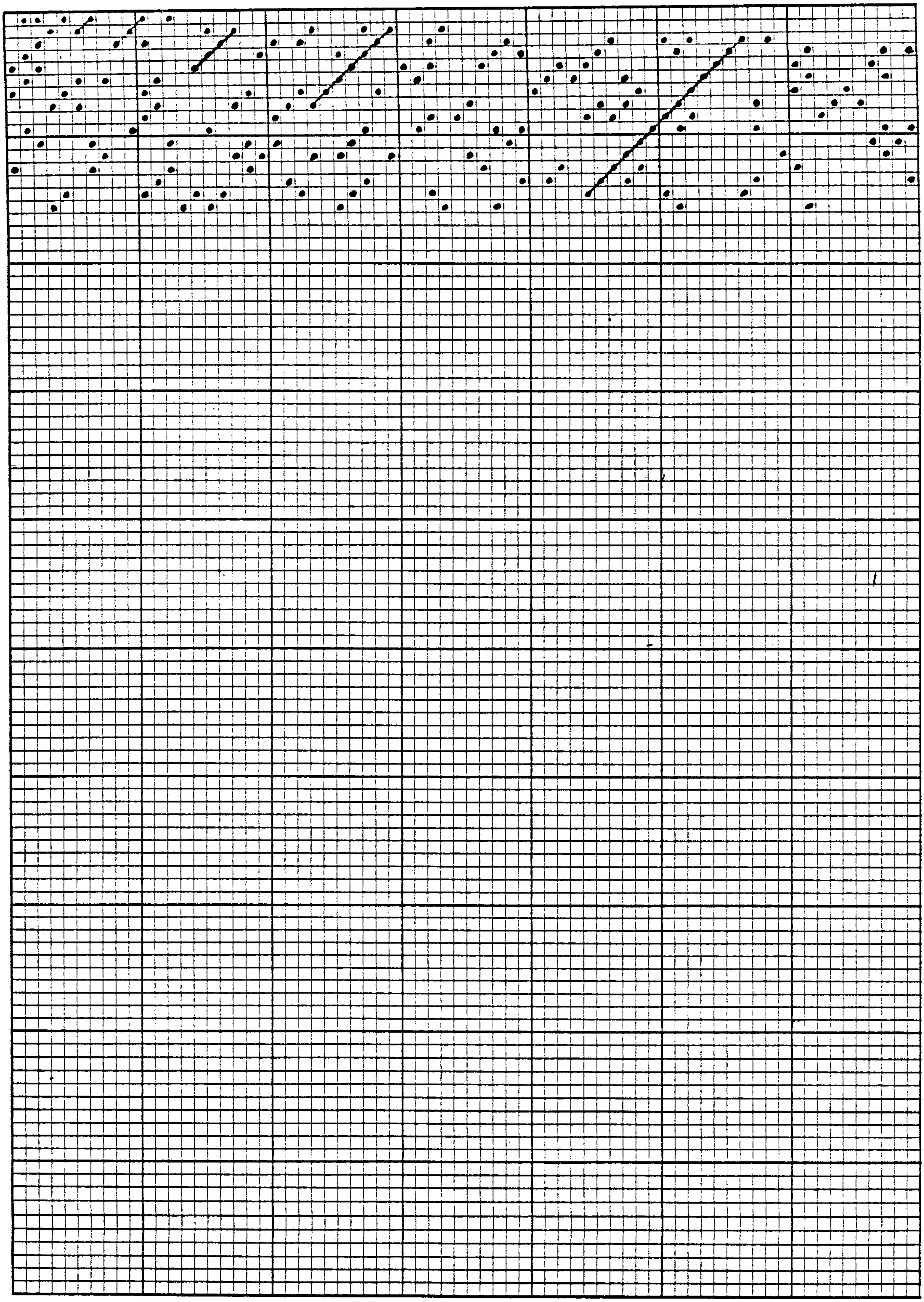

Figure 125.

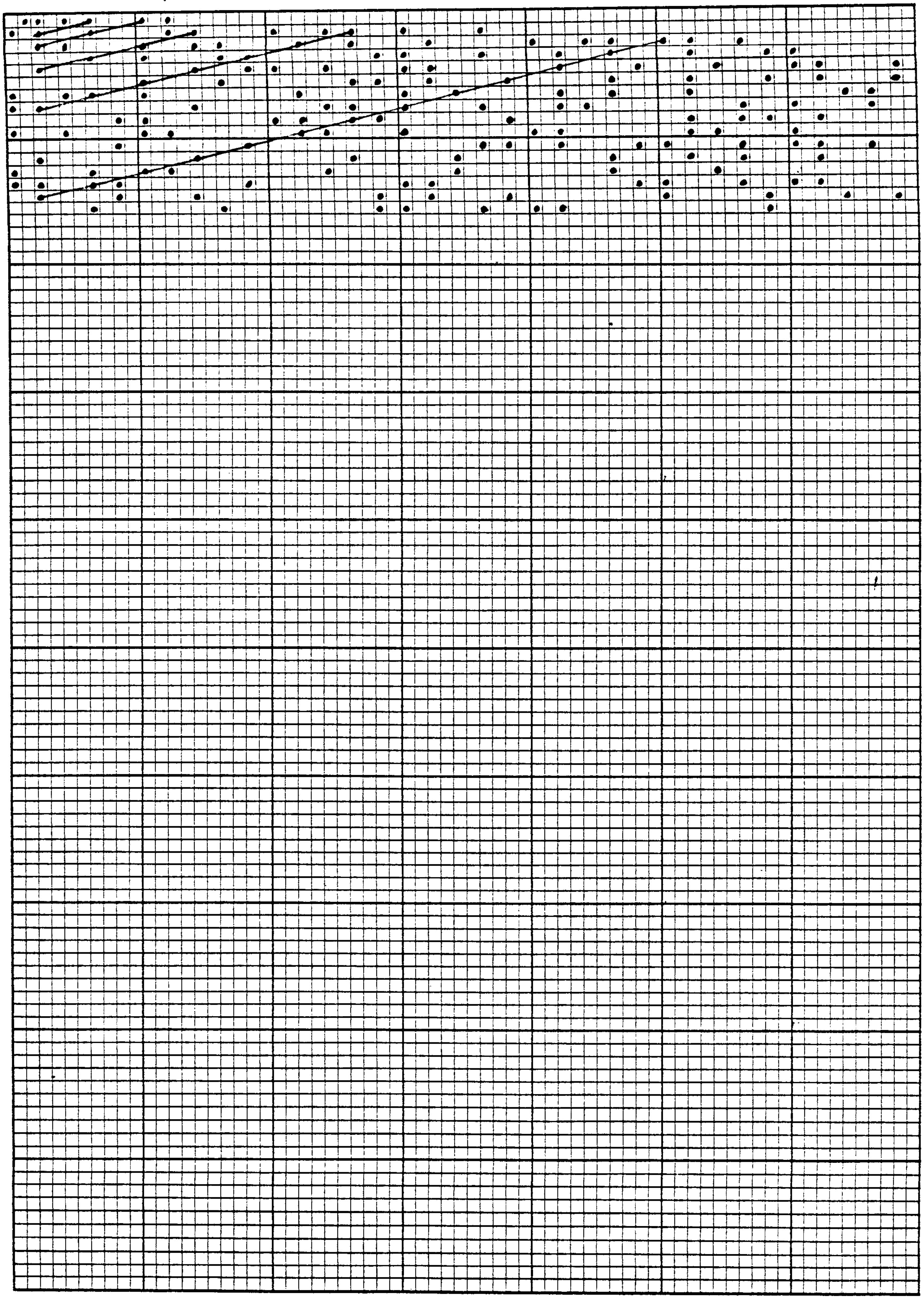

Figure 136.

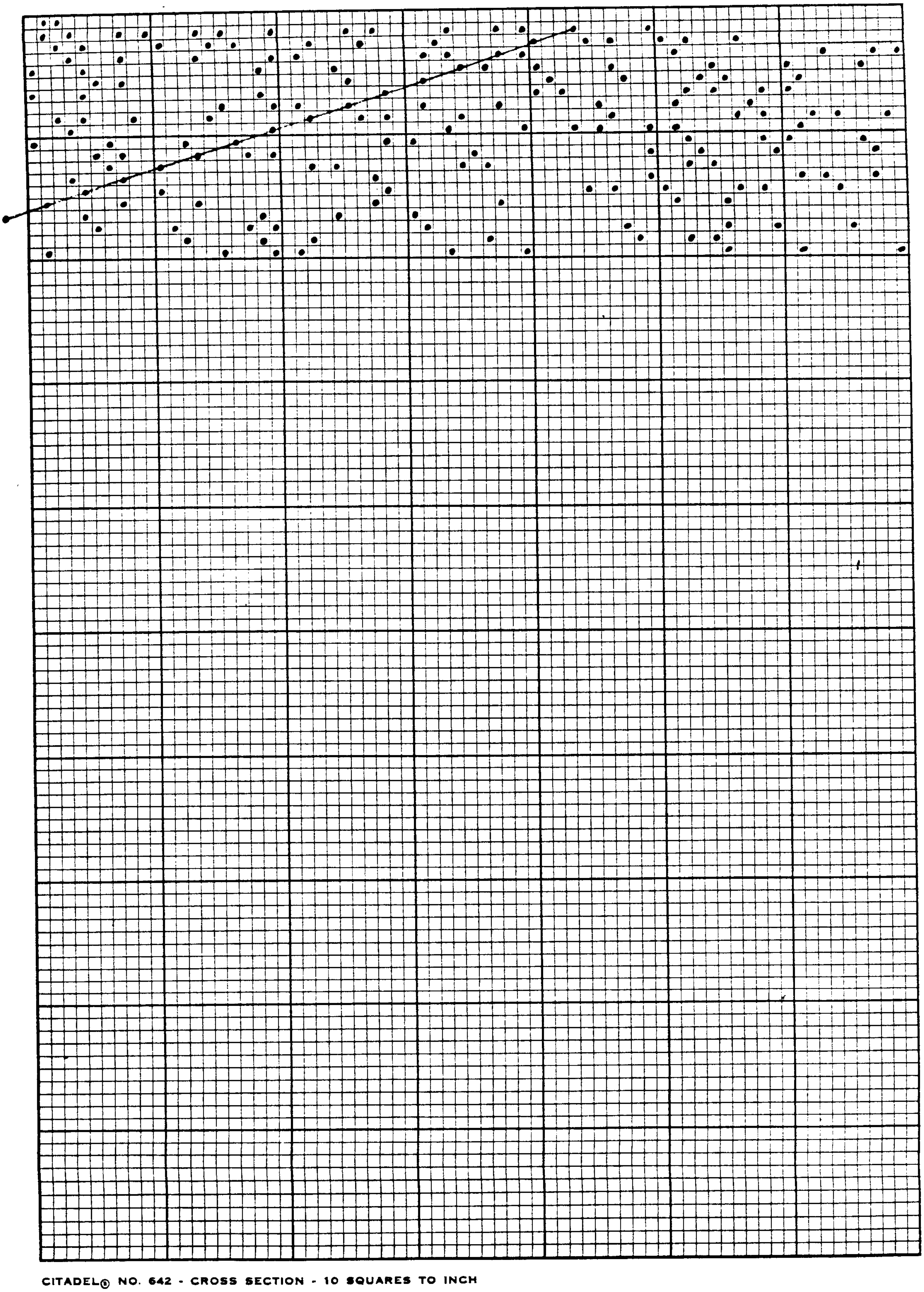

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Figure 127.

STRETCHED AND ANY OF THEM HITS SUCH A BARRIER, ALL WILL HIT IT. AN EXAMPLE OF THIS IS THE BEHAVIOR OF THE "HALF-DIAGONALS" GENERATED BY

$$24X^2 + 24X + 13$$
$$24X^2 + 24X + 19$$
$$\text{AND} \quad 24X^2 + 24X + 23$$

WHEN STRETCHED BY ARRAY JUT-48 TOP-2. THE BARRIER COLUMN IS COLUMN 6 (SEE FIGURE 128).

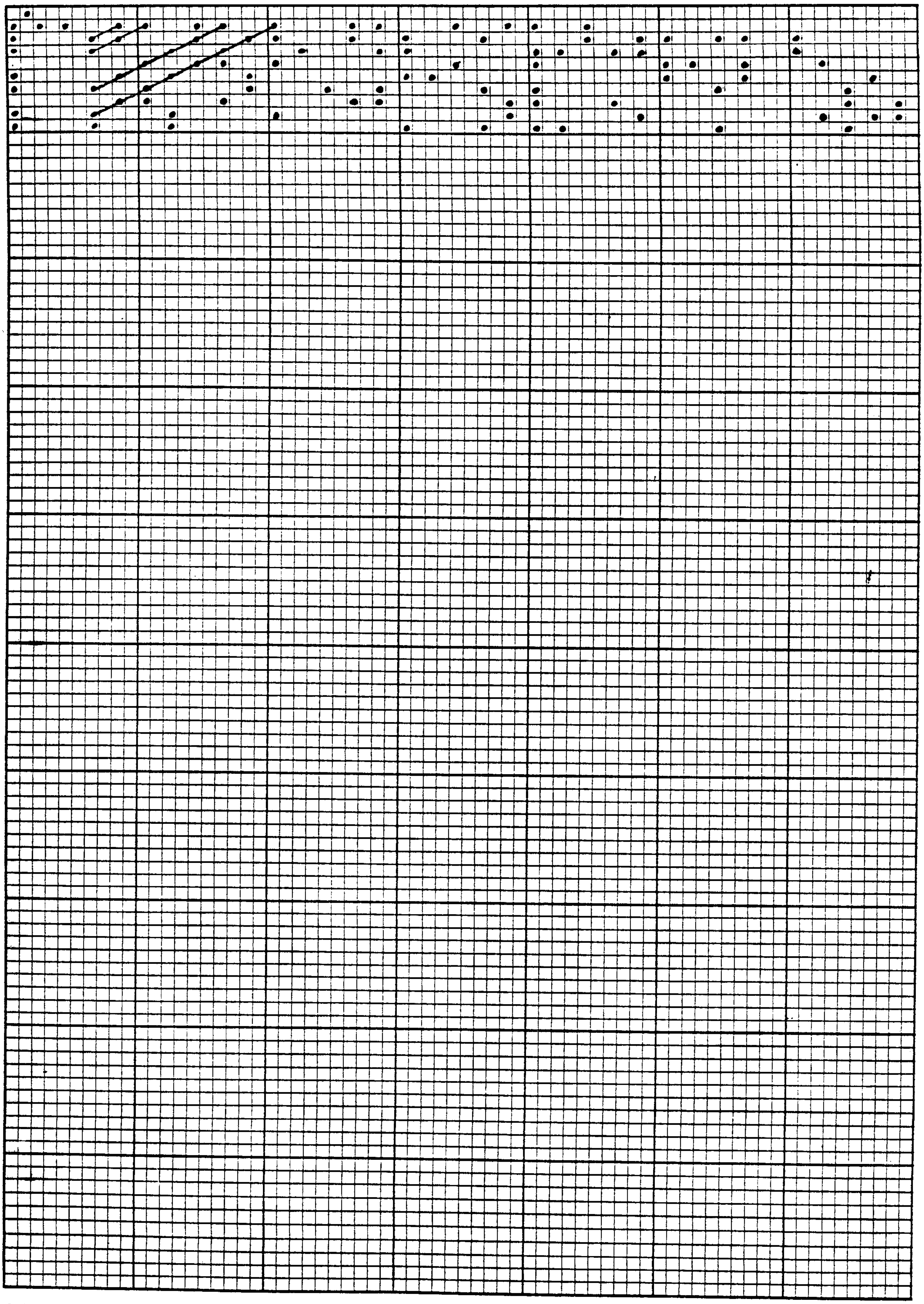

Figure 128.

MORE PATTERNS WITH LINE CONTINUATION

$6x^2 + 6x + 5$	3 3
$6x^2 + 5$	5 3 2
$6x^2 + 6x + 7$	5 3 2
$6x^2 + 7$	7 3 2 1 1
$6x^2 + 6x + 11$	9 3 2 1 1 1
$6x^2 + 13$	13 3 2 1 1 1 4
$6x^2 + 6x + 17$	15 3 2 1 1 1 4 1 2
$6x^2 + 17$	17 3 2 1 1 1 4 1 2 1
$6x^2 + 6x + 31$	29 3 2 1 1 1 4 1 2 1 6 1 3 1

+2 +4 +6 +14

X	$6X^2+6X$	+5	+7	+11	+17	+31
0	0	5	7	11	17	31
1	12	17	19	23	29	43
2	36	41	43	47	53	67
3	72	773	79	83	89	103
4	120	125	127	131	137	151
5	180	185	187	191	197	211
6	252	257	259	263	269	283
7	336	341	343	347	353	367
8	432	437	439	443	449	463
9	540	545	547	551	557	571
10	660	665	667	671	677	691
11	792	797	799	803	809	823
12	936	941	943	947	953	967
13	1092	1097	1099	1103	1109	1123
14	1260	1265	1267	1271	1277	1291
15	1440	1445	1447	1451	1457	1471
16	1632	1637	1639	1643	1649	1663
17	1836	1841	1843	1847	1853	1867
18	2052	2057	2059	2063	2069	2083
19	2280	2285	2287	2291	2297	2311
20	2520	2525	2527	2531	2537	2551
21	2772	2777	2779	2783	2789	2803
22	3036	3041	3043	3047	3053	3067
23	3312	3317	3319	3323	3329	3343
24	3600	3605	3607	3611	3617	3631
25	3900	3905	3907	3911	3917	3931
26	4212	4217	4219	4223	4229	4243
27	4536	4541	4543	4547	4553	4567

+31 CT'D

28	4872	4903
29	5220	5251
30	5580	5611
31	5952	5983
32	6336	6367
33	6732	6763
34	7140	7171
35	7560	7591
36	7992	8023
37	8436	8467
38	8892	8923
39	9360	93[illegible]1
40	9840	9871
41	10332	10363
42	10836	10867
43	11352	11383
44	11880	11911
45	12420	12451
46	12972	13003
47	13536	13567
48	14112	14143
49	14700	14731
50	15300	15331
51	15912	15943
52	16536	16567
53	17172	17203
54	17820	17851
55	18480	18511
56	19152	19183

			+2	+6	+4
x	$6x^2$	+5	+7	+13	+17
0	0	5	7	13	17
1	6	11	13	19	23
2	24	29	31	37	41
3	54	59	61	67	71
4	96	101	103	109	113
5	150	155	157	163	167
6	216	221	223	229	233
7	294	299	301	307	311
8	384	389	391	397	401
9	486	491	493	499	503
10	600	605	607	613	617
11	726	731	733	739	743
12	864	869	871	877	881
13	1014	1019	1021	1027	1031
14	1176	1181	1183	1189	1193
15	1350	1355	1357	1363	1367
16	1536	1541	1543	1549	1553
17	1734	1739	1741	1747	1751
18	1944	1949	1951	1957	1961
19	2166	2171	2173	2179	2183
20	2400	2405	2407	2413	2417
21	2646	2651	2653	2659	2663
22	2904	2909	2911	2917	2921
23	3174	3179	3181	3187	3191
24	3456	3461	3463	3469	3473
25	3750	3755	3757	3763	3767
26	4056	4061	4063	4069	4073
27	4374	4379	4381	4387	4391

X	$6X^2+6X+17$ CT'D	
28	4889	348
29	5237	360
30	5597	372
31	5969	384
32	6353	396
	6749	408
	7157	420
	7577	432
	8009	444
	8453	

$6X^2+17$ CT'D	
4721	342
5063	354
5417	366
5783	378
6161	390
6551	402
6953	414
7367	426
7793	438
8231	

X	$6X^2+6X+31$ CT'D	
57	19867	
58	20563	
59	21271	720
60	21991	732
61	22723	744
62	23467	756
63	24223	768
64	24991	780
65	25771	792
66	26563	

x	x^2	$6x^2$	$6x$	$6x^2 + 6x$	x	x^2	$6x^2$	$6x$	$6x^2 + 6x$
0	0	0	0	0	26	676	4056	156	4212
1	1	6	6	12	27	729	4374	162	4536
2	4	24	12	36	28	784	4704	168	4872
3	9	54	18	72	29	841	5046	174	5220
4	16	96	24	120	30	900	5400	180	5580
5	25	150	30	180	31	961	5766	186	5952
6	36	216	36	252	32	1024	6144	192	6336
7	49	294	42	336	33	1089	6534	198	6732
8	64	384	48	432	34	1156	6936	204	7140
9	81	486	54	540	35	1225	7350	210	7560
10	100	600	60	660	36	1296	7776	216	7992
11	121	726	66	792	37	1369	8214	222	8436
12	144	864	72	936	38	1444	8664	228	8892
13	169	1014	78	1092	39	1521	9126	234	9360
14	196	1176	84	1260	40	1600	9600	240	9840
15	225	1350	90	1440	41	1681	10086	246	10332
16	256	1536	96	1632	42	1764	10584	252	10836
17	289	1734	102	1836	43	1849	11094	258	11352
18	324	1944	108	2052	44	1936	11616	264	11880
19	361	2166	114	2280	45	2025	12150	270	12420
20	400	2400	120	2520	46	2116	12696	276	12972
21	441	2646	126	2772	47	2209	13254	282	13536
22	484	2904	132	3036	48	2304	13824	288	14112
23	529	3174	138	3312	49	2401	14406	294	14700
24	576	3456	144	3600	50	2500	15000	300	15300
25	625	3750	150	3900					

564
12
576
12
588
12
600

15912 612
16536 624
17172 636
17820 648
18480 660

18480 672
19152
19836 684
20532 696
21240 708

+	−
40	2
2	1
4	1
6	1
8	1
10	1
4	2
1	1
2	1
1	1
1	1
4	1
5	1
1	1
4	1
7	1
3	3
2	2
2	1
5	1
1	1
1	1
2	1
3	1
7	1
3	1
1	4

+	−
5	1
1	2
4	1
5	4
2	1
10	1
2	2
1	3
3	1
1	4
13	1
1	1
1	2
1	1
2	1
1	3
1	2
1	2
2	2
2	1
1	1
3	2
1	1
1	2
6	2
2	2
1	2

+	−
1	1
1	1
3	2
1	5
6	1
2	1
1	1

x^2+x+41

370	34451
372	34823
374	35197
376	35573
378	35951
380	36331
382	36713
384	37097
386	37483
388	37871
390	38261
392	38653
394	39047
396	39443
398	39841
400	40241
402	40643
404	41047
406	41453
408	41861
410	42271
412	42683
414	43097
416	43513
418	43931
420	44351
422	44773
424	45197
426	45623
428	46051
430	46481
432	46913
434	47347
436	47783
438	48221
440	48661
442	49103
444	49547
446	49993
448	50441
450	50891
452	51343
454	51797
456	52253
458	52711
460	53171
462	53633
464	54097
466	54563
468	55031
470	55501
472	55973
474	56447
476	56923
478	57401
480	57881
482	58363
484	58847
486	59333
488	59821
490	60311
492	60803
494	61297
496	61793
498	62291
500	62791
502	63293
504	63797
506	64303
508	64811
510	65321
512	65833
514	66347
516	66863
518	67381
520	67901
522	68423
524	68947
526	69473
528	70001
530	70531
532	71063
534	71597
536	72133
538	72671 ~~[illegible]~~
540	73211
542	73753
544	74297
546	74843
548	75391
550	75941
552	76493
554	77047
556	77603
558	78161
560	78721
562	79283
564	79847
566	80413
568	80981
570	81551
572	82123
574	82697
576	83273
578	83851
580	84431
582	85013
584	85597
586	86183
588	86771
590	87361
592	87953

over

40	1	-4
-2	4	5
2	1	1
1	7	1
4	1	2
1	3	4
6	-3	1
1	2	5
8	2	⋮
1	2	
10	1	
1	5	
4	1	
-2	1	
1	1	
1	1	
2	1	
1	2	
1	1	
1	3	
1	1	
1	7	
4	1	
1	3	
5	1	
1	1	
1	[illegible]	

LINE CONTINUATION

X	X^2+X+41	
76	5893	
77	6047	154
78	6203	156
79	6361	158
80	6521	160
	6683	162
	6847	164
	7013	166
	7181	168
	7351	170
	7523	172
	7697	174
	7873	176
	8051	178
90	8231	180
	8413	182
	8597	184
	8783	186
	8971	188
	9161	190
	9353	192
	9547	194
	9743	196
	9941	198
100	10141	200
	10343	202
	10547	204

10753	206
10961	208
11171	210
11383	212
11597	214
11813	216
12031	218
12251	220
12473	222
12697	224
12923	226
13151	228
13381	230
13613	232
13847	234
14083	236
14321	238
14561	240
14803	242
15047	244
15293	246
15541	248
15791	250
16043	252
16297	254
16553	256
16811	258

17071	260
17333	262
17597	264
17863	266
18131	268
18401	270
18673	272
18947	274
19223	276
19501	278
19781	280
20063	282
20347	284
20633	286
20921	288
21211	290
21503	292
21797	294
22093	296
22391	298
22691	300
22993	302
23297	304
23603	306
23911	308
24221	310
24533	312

24847	314
25163	316
25481	318
25801	320
26123	322
26447	324
26773	326
27101	328
27431	330
27763	332
28097	334
28433	336
28771	338
29111	340
29453	342
29797	344
30143	346
30491	348
30841	350
31193	352
31547	354
31903	356
32261	358
32621	360
32983	362
33347	364
33713	366
34081	368